Signal & Systems

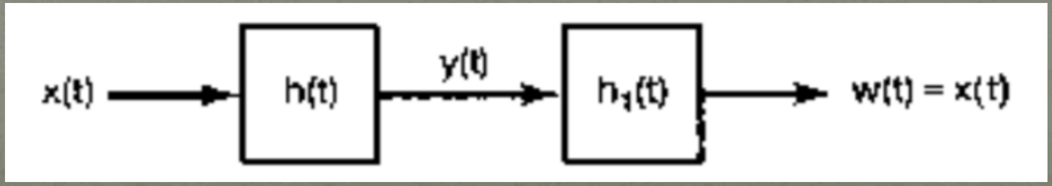
Lecture # 5
Properties of LTI Systems

22nd October 18

Properties

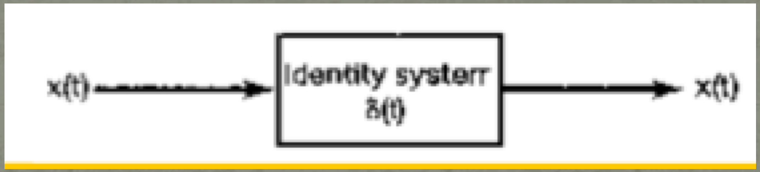
Invertibility of LTI Systems

- A system is invertible if and only if an inverse system exists that, when connected in series with the original system, produces an output equal to the input to the first system.
- If an LTI system is invertible, then it has an LTI inverse.
- Suppose we have a system with impulse response h(t). The inverse system with impulse response $h_1(t)$ results in w(t)=x(t) as shown below in series interconnection.



Invertibility of LTI Systems (cont.)

- This system is identical to the identity system shown below:



- Since the overall impulse response shown in first figure is $h(t)*h_1(t)$, we have the condition that $h_1(t)$ must satisfy for it to be the impulse response of the inverse system, i.e., $h(t)*h_1(t)=\delta(t)$
- Similarly in discrete time the impulse response $h_1[n]$ of the inverse system for an LTI system with impulse response $h[n]$ must satisfy: $h[n]*h_1[n]=\delta[n]$

Example #1

- Consider the LTI system consisting of a pure time shift:

$$y(t) = x(t - t_0)$$

Causality of LTI Systems

- The output of a causal system depends only on the present and past values of the input to the system.
- Theorem: An LTI system is causal if and only if:

$$h[n]=0, \quad for \quad all \quad n<0 \qquad h(t)=0, \quad for \quad all \quad t<0$$

- Proof: If S is causal, then the output y[n] cannot depend on x[k] for k>n.
- From the convolution equation:

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- We must have:

$$h[n-k]=0, \quad for \quad k>n$$

Causality of LTI Systems (cont.)

- Or equivalently $h[n-k]=0, \quad for \quad n-k<0$
- Setting m = n – k, we see that: h[m] =0, for m < 0.
- Conversely, if h[k] = 0 for k < 0, then for input x[n],

$$y[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]=\sum_{k=0}^{\infty} h[k]x[n-k]$$

- Therefore, y[n] depends only upon x[m] for m ≤ n.

Example #2

- Check whether the following system is causal or not:
  - (1): $h(t) = 3\delta(t+2)$
  - (2): $h(t) = e^{-(t+1)} \cdot u(t)$

Stability of LTI Systems

- A system is stable if every bounded input produces a bounded output.
- In order to determine conditions under which LTI systems are stable, consider an input x[n] that is bounded in magnitude:

$$\left|x[n]\right| < B, \quad for \quad all \quad n$$

- Suppose that this input is applied to an LTI system with unit impulse response h[n].
- Then, using the convolution sum, we obtain an expression for the magnitude of the output:

$$\left|y[n]\right| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right|$$

Stability of LTI Systems (cont.)

- Since the magnitude of the sum of a set of numbers is no longer than the sum of the magnitudes of the numbers, it follows from above equation that:

$$\left|y[n]\right| \leq \sum_{k=-\infty}^{\infty} \left|h[k]\right| \left|x[n-k]\right|$$

- As $\left|x[n-k]\right| < B$ for all values of k and n, then:

$$\left|y[n]\right| \leq B \sum_{k=-\infty}^{\infty} \left|h[k]\right|, \quad for \quad all \quad n$$

- From above equation we can conclude that if the impulse response is absolutely summable that is, if:

$$\sum_{k=-\infty}^{\infty} \left|h[k]\right| < \infty$$

Stability of LTI Systems (cont.)

- Then y[n] is bounded in magnitude and hence the system is stable.
- If above equation is not satisfied, there are bounded inputs that result in unbounded outputs.
- Thus in continuous time case the system is stable if the impulse response is absolutely integrable, i.e., if:

$$\int_{-\infty}^{\infty} \left| h(\tau) \right| d\tau < \infty$$

LTI Systems Described by Differential and Difference Equations

Linear Constant-Coefficient Differential Equations (cont.)

- A general Nth-order linear constant-coefficient differential equation is given by:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- Where coefficients a_k and b_k are real constants.
- The order N refers to the highest derivative of y(t).
- The general solution of above equation for a particular input x(t) is given by:

$$y(t) = y_p(t) + y_h(t)$$

Linear Constant-Coefficient Differential Equations (cont.)

- Where $y_p(t)$ is a particular solution.
- $y_h(t)$ is a homogeneous solution, satisfying the homogeneous differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y_h(t)}{dt^k} = 0$$

Causality of LCCDE

- For the linear system to be causal we must assume the condition of initial rest, i.e., if x(t)=0 for $t \le t_o$, then assume y(t)=0 for $t \le t_o$.
- Thus, the response for $t > t_o$ can be calculated with the initial conditions:

$$y(t_0) = \frac{dy(t_0)}{dt} = \ldots = \frac{d^{N-1}y(t_0)}{dt^{N-1}} = 0$$

where

$$\frac{d^k y(t_0)}{dt^k} = \left.\frac{d^k y(t)}{dt^k}\right|_{t=t_0}$$

Linear Constant-Coefficient Difference Equations

- The Nth-order linear constant coefficient difference equation is: $$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$
- The solution y[n] can be written as the sum of a particular solution and a solution to the homogeneous equation is:

$$\sum_{k=0}^{N} a_k y[n-k] = 0$$

- The solution to this homogeneous equations are often referred to as the natural responses of the system.

Linear Constant-Coefficient Difference Equations

- For auxiliary conditions we will focus on the condition of initial rest.
- That is if x[n]=0 for $n < n_0$, then y[n]=0 for $n < n_0$.
- With initial rest the system is LTI and causal.
- The above equation can be rearranged in the form:

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\}$$

- These equations are known as recursive equation.
- In the special case when N=0, the above equation reduces to:

$$y[n] = \sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)x[n-k]$$

Linear Constant-Coefficient Difference Equations

- Here y[n] is an explicit function of the present and previous values of the input.
- Above equation is also known as non-recursive equation.
- The above equation describes an LTI system and by direct computation, the impulse response of this system is found to be:

$$h[n] = \begin{cases} \dfrac{b_n}{a_0}, & 0 \le n \le M \\ 0, & otherwise \end{cases}$$

- The above equation is nothing more than the convolution sum.

Example #3

- Consider the difference equation:

$$y[n]-\frac{1}{2}y[n-1]=x[n]$$

The End