Signal & Systems

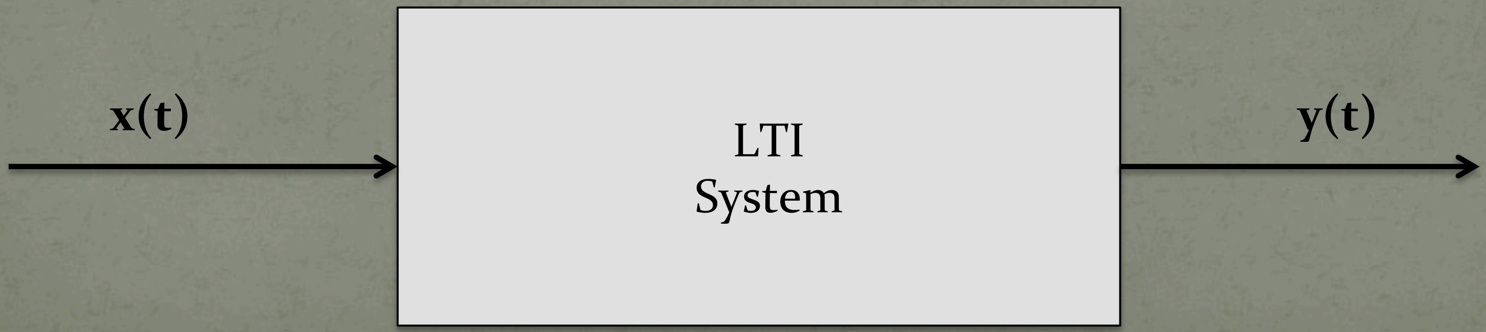
Lecture # 4
Convolution

8th November 18

LTI Systems

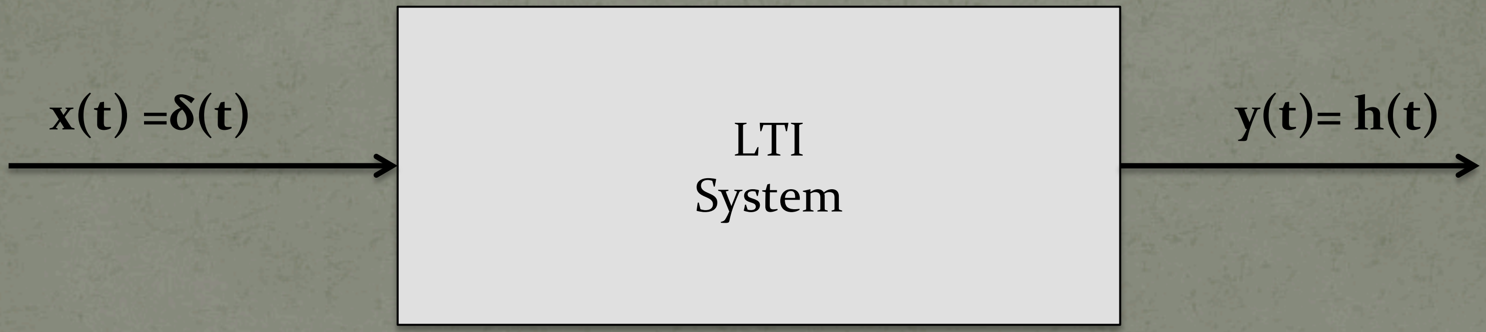
LTI Systems

- LTI= Linear Time Invariant Systems
- As Linear systems follow the principle of superposition, hence LTI will also follow the principle of superposition.
- Any delay in the input is reflected in output, property of Time invariant system will also be followed by LTI system.



Impulse Response

- Impulse response is the output of an LTI system and it is only related to LTI systems.
- Unit impulse signal is the input of LTI system and impulse response h(t) is the output or response of unit impulse signal.



Introduction to Convolution

Introduction

- To study LTI systems, linear convolution plays an important role.
- Majority of discrete time systems in practice are shift invariant and linear and in many cases we take continuous systems as linear systems.
- The input signal can be decomposed into a set of impulses, each of which is scales and shifted delta/impulse function.
- The output from each impulse is scaled and shifted version of the impulse response.
- Then the overall output signal can be added to form one output.

Introduction (cont.)

- That is if we know the system's impulse response we can calculate the output for any possible input.
- This response is known as convolution kernel.
- We use " * " as the convolution operator.
- We convolve two signals and produces the third signal.
- Convolution is basically a relationship between three signals i-e; input signal, the impulse response and the output signal.

Convolution Integral

Continuous-Time Convolution

- In continuous time signal the signal decomposition is:

$$y(t) = T[x(t)] = T\left[\sum_{n=-\infty}^{\infty} x(\tau)\delta(n-\tau)\right]$$

- The continuous time convolution is defined as:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau = h(t) * x(t)$$

Convolution Theorem

- h(t) is the impulse response of an analog or continuous time system for an input δ(t).
- Hence for a Linear system its output for a shifted impulse δ(t-T) is h(t-T).
- Therefore,

$$\delta(t) \rightarrow h(t)$$

$$\delta(t-T) \rightarrow h(t-T)$$

$$x(t)\delta(t-T) \rightarrow x(T)h(t-T)$$

- Obviously,

$$\int_{-\infty}^{\infty} x(T)\delta(t-T)\,dT \rightarrow \int_{-\infty}^{\infty} x(T)h(t-T)\,dT$$

Convolution Theorem (cont.)

- Therefore, we can write by the linearity of the system:

$$y(t) = \int_{-\infty}^{\infty} x(T)h(t-T)\,dT$$

- This is the convolution of x(t) with h(t).

$$\therefore y(t) = \quad Convolution \quad of \quad h(t) \quad \& \quad x(t)$$

$$= x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(T)h(t-T)\,dT$$

Example #1

- The input x(t) and the impulse response h(t) of a continuous time LTI system are given by:

$$x(t) = u(t)$$

$$h(t) = e^{-at}u(t), \quad a > 0$$

- Compute the output y(t).

Step Response

- The step response s(t) of continuous-time LTI system is defined to be the response of the system when the input is u (t); i.e.,

$$s(t) = T\{u(t)\}$$

- The step response s(t) can be easily determined by:

$$s(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau) u(t-\tau) d\tau = \int_{-\infty}^{t} h(\tau) d\tau$$

- The step response s(t) can be obtained by integrating the impulse response h(t). Differentiating above equation with respect to t, we can determine the impulse response; i.e.,

$$h(t) = s'(t) = \frac{ds(t)}{dt}$$

Steps for Graphical Convolution

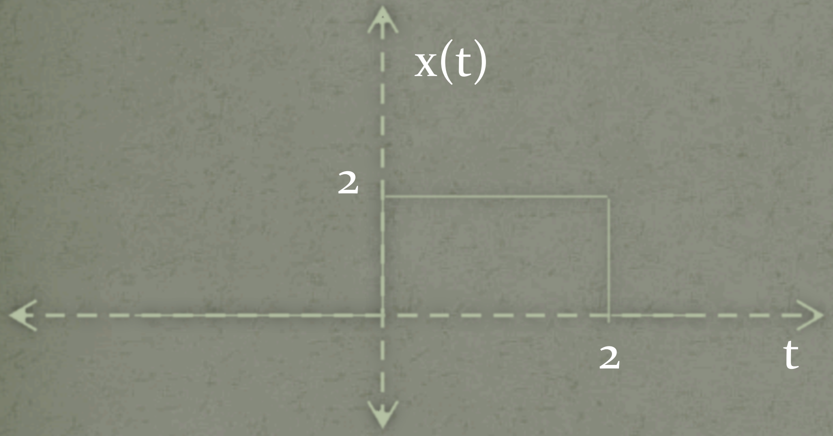
- Steps for Graphical convolution are:
  - Re-write the signals as functions of τ: $x(\tau)$ & $h(\tau)$.
  - Flip one of the signals around t=0 to get either $x(-\tau)$ or $h(-\tau)$.
    - It is best to flip the signal with shorter duration.
    - We'll flip $h(\tau)$ to get $h(-\tau)$ here, for notational purposes.
  - Find edges of the flipped signal.
    - Find the left-hand edge of τ value of $h(-\tau)$: call it $\tau_{L,0}$
    - Find the right hand edge τ value of $h(-\tau)$: call it $\tau_{R,0}$
  - Shift $h(-\tau)$ by an arbitrary value of t to get $h(t-\tau)$ and get its edges.
    - Find the left-hand edge of τ value of $h(t-\tau)$ as a function t: call it $\tau_{L,t}$ it will always be $\tau_{L,t}=t+\tau_{L,t}$
    - Find the right hand edge τ value of $h(t-\tau)$ as a function t: call it $\tau_{R,t}$ it will always be $\tau_{R,t}=t+\tau_{R,t}$

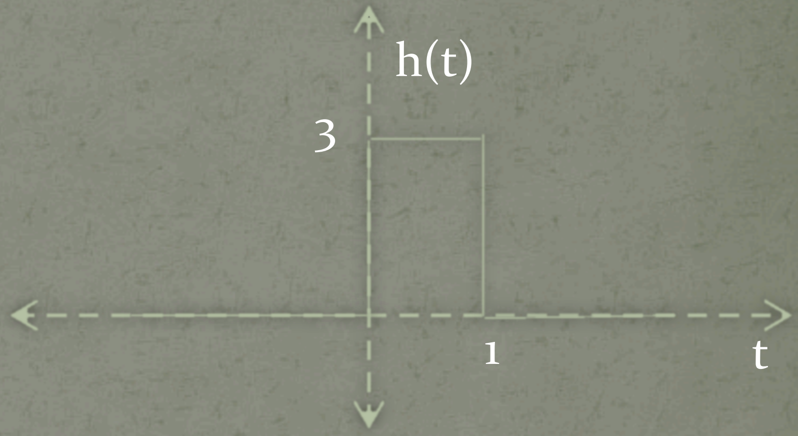
Steps for Graphical Convolution (cont.)

- Find regions of τ-Overlap:
  - Find intervals of t over which the product x(τ)h(t-τ) has a single mathematical form in terms of τ.
  - In each region find interval of t that makes the identified overlap happen.
- For each region form the product of x(τ)h(t-τ) and integrate.
- Assemble the output from the output time-sections for all the regions.

Example #2

- Evaluate y(t) = x(t) * h(t) by using graphical method, where x(t) and h(t) are shown below.

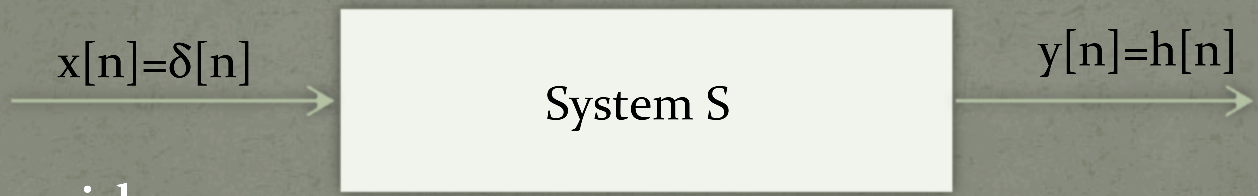




Convolution Sum

Representation of Discrete-Time Signals in Terms of Impulses

- Denote by h[n] the "impulse response" of an LTI system S.
- The impulse response is the response of the system to a unit impulse input.
- The definition of an unit impulse is: $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$



- Let consider:

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

Representation of Discrete-Time Signals in Terms of Impulses (cont.)

- Using the above fact we get the following equalities:

$$x[n]\delta[n] = x[0]\delta[n], \quad (n_0 = 0)$$

$$x[n]\delta[n-1] = x[1]\delta[n-1], \quad (n_0 = 1)$$

$$x[n]\delta[n-2] = x[2]\delta[n-2], \quad (n_0 = 2)$$

$$\vdots \qquad\qquad \vdots$$

$$= x[n]\left(\sum_{k=-\infty}^{\infty} \delta[n-k]\right) \qquad = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- The sum on the left hand side is:

$$x[n]\left(\sum_{k=-\infty}^{\infty} \delta[n-k]\right) = x[n]$$

- Because $\sum_{k=-\infty}^{\infty} \delta[n-k] = 1$ for all n.

Representation of Discrete-Time Signals in Terms of Impulses (cont.)

- The sum on the right hand side is:

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- Therefore, equating the left hand side and right hand side yields:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- Any signal x[n] can be expressed as a sum of impulses.
- Suppose we know that the impulse response of an LTI system is h[n] and we want to determine the output y[n].
- To do so we first express x[n] as a sum of impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Representation of Discrete-Time Signals in Terms of Impulses (cont.)

- For each impulse δ [n-k], we can determine its impulse response, because for an LTI system:

$$\delta[n-k] \rightarrow h[n-k]$$

- Consequently, we have:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow \sum_{k=-\infty}^{\infty} x[k]h[n-k] = y[n]$$

- Where the equation: $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$ is known as the convolution equation.

Step Response

- The step response s[n] of a discrete-time LTI system with the impulse response h[n] is as follows:

$$s[n] = h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{n} h[k]$$

- Then we have:

$$h[n] = s[n] - s[n-1]$$

Evaluating Convolution

- There are three basic steps used to evaluate convolution of any signal:
  - Flip
  - Shift
  - Multiply and Add.

Example #3

- Compute y[n] = x[n] * h[n] using analytical method, where:

$$x[n] = \alpha^n u[n] \quad , \quad h[n] = \beta^n u[n]$$

Example #4

- Consider the signal x[n] and the impulse response h[n] shown below:

Properties of LTI Systems

Commutative Property

- A basic property of convolution in both continuous and discrete time is that it is a commutative operation.
- In discrete time it is:

$$x[n]*h[n]=h[n]*x[n]=\sum_{k=-\infty} h[k]x[n-k]$$

- In continuous time it is:

$$x(t)*h(t)=h(t)*x(t)=\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Proof: In the discrete time case if we let r=n-k or equivalently k=n-r then:

$$x[n]*h[n]=\sum_{k=-\infty}^{+\infty} x[k]h[n-k]=\sum_{r=-\infty}^{+\infty} x[n-r]h[r]=h[n]*x[n]$$

- With this substitution of variables, the roles of x[n] and h[n] are interchanged.

Commutative Property (cont.)

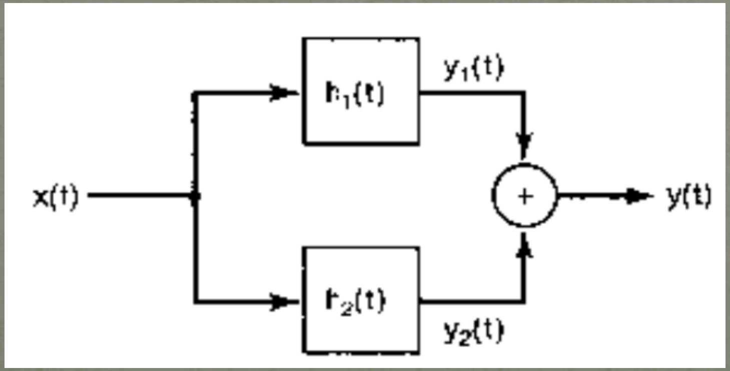
- This property states that one of the two forms for computing convolutions in discrete time and continuous time may be easier to visualize, but both forms always result in the same answer.

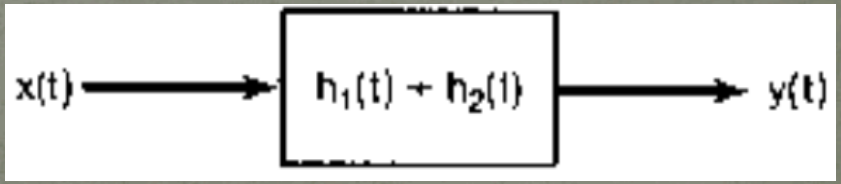
Distributive Property

- Convolution distributes over addition so that in discrete time: $x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+x[n]*h_2[n]$
- In continuous time:

$$x(t)*[h_1(t)+h_2(t)] = x(t)*h_1(t)+x(t)*h_2(t)$$

- Interpretation of Distributive property of convolution for a parallel interconnection of LTI systems is shown below:





Distributive Property (cont.)

- The two systems with impulse responses $h_1(t)$ and $h_2(t)$ have identical inputs and their outputs are added.
- Since:

$$y_1(t) = x(t) * h_1(t)$$

$$and$$

$$y_2(t) = x(t) * h_2(t)$$

- The system of above figure has output:

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

- The system of second figure has the output:

$$y(t) = x(t) * \left[h_1(t) + h_2(t)\right]$$

- Comparing both the above results we see that the systems in above figures are identical.
- In same way distributive property of discrete time can also be proved.

Associative Property

- Another important and useful property of convolution is associative property.
- In discrete time: $x[n]*(h_1[n]*h_2[n]) = (x[n]*h_1[n])*h_2[n]$
- In continuous time: $x(t)*[h_1(t)*h_2(t)] = [x(t)*h_1(t)]*h_2(t)$

- As a consequence of the associative property, the expressions:

$$y[n] = x[n]*h_1[n]*h_2[n]$$

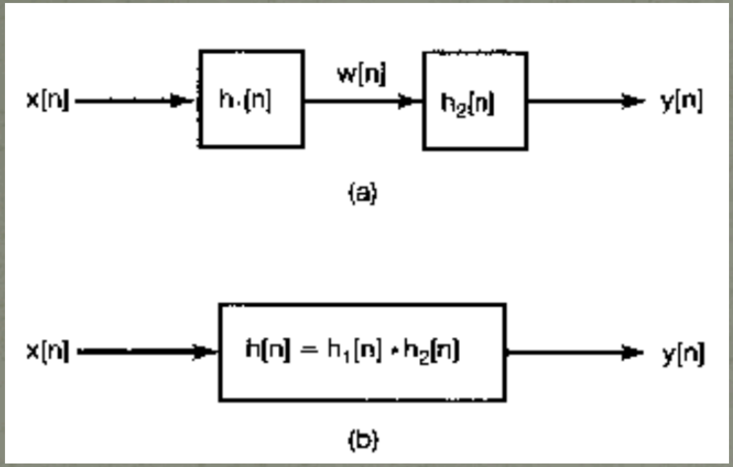
and

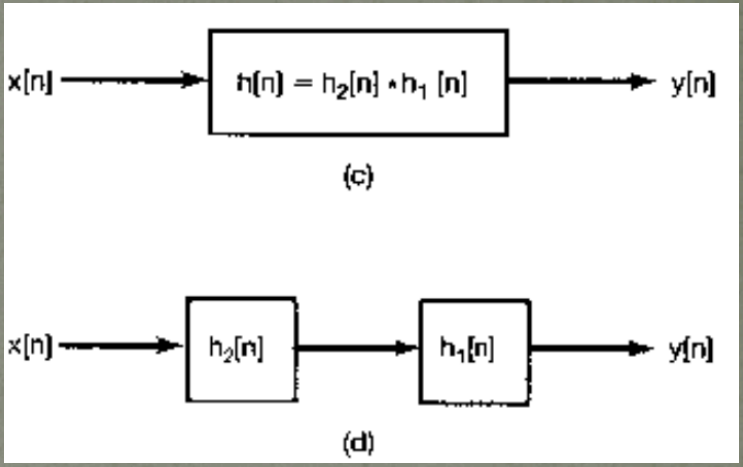
$$y(t) = x(t)*h_1(t)*h_2(t)$$

- Are unambiguous. That is it does not matter in which order we convolve these signals.

Associative Property (cont.)

- An interpretation of the associative property is illustrated for discrete time systems in figures below:





$$y[n] = w[n] * h_2[n] = \left(x[n] * h_1[n]\right) * h_2[n], \quad fig(a)$$

$$y[n] = x[n] * h[n] = x[n] * \left(h_1[n] * h_2[n]\right), \quad fig(b)$$

Associative Property (cont.)

- According to the associative property the series interconnection of the two systems in fig(a) is equivalent to the single system in fig(b).
- This can be generalized to an arbitrary number of LTI systems in cascade and the analogous interpretation and conclusion also hold in continuous time.
- By using the commutative property together with the associative property, we find another very important property of LTI systems.
- From fig(a) and (b) we can conclude that the impulse response of the cascade of two LTI system is the convolution of their individual impulse responses.
- Since convolution is commutative we can compute this convolution of $h_1[n]$ and $h_2[n]$ in either order.

LTI Systems With & Without Memory

- A system is memory less if the output depends on the current input only.
- An LTI system is memory less if and only if h[n] = 0 for n≠0.
- In this case the impulse response has the form: h[n] = K δ[n] where K = h[0] is a constant and the convolution sum reduces to the relation:

$$y[n] = Kx[n]$$

- If a discrete time LTI system has an impulse response h[n] that is not identically zero for n≠0, then the system has memory.
- A continuous time LTI system is memory less if h(t) = 0 for t ≠ 0, and such a memory less LTI system has the form: y(t) =K x(t) for some constant K and has the impulse response:

$$h(t) = K\delta(t)$$

LTI Systems With & Without Memory (cont.)

- Note that if K=1 in discrete and continuous time impulse response then these systems become identity systems with output equal to the input and with unit impulse response equal to the unit impulse.
- In this case the convolution sum and integral formulas imply that:

$$x[n] = x[n] * \delta[n]$$

$$\text{and}$$

$$x(t) = x(t) * \delta(t)$$

- Which reduces to the sifting properties of the discrete and continuous time unit impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

The End