15th Oct, 18

-: Lectur # 4 :-

"Convolution"

Example # 1 :-

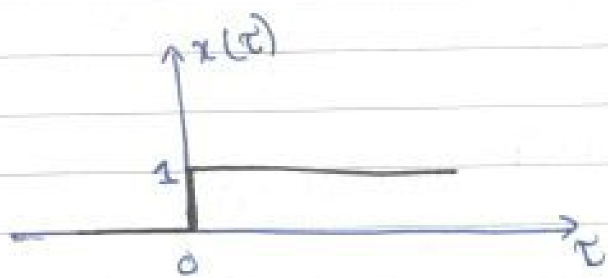
$x(t) = u(t)$, $h(t) = e^{-at} u(t)$, $a > 0$

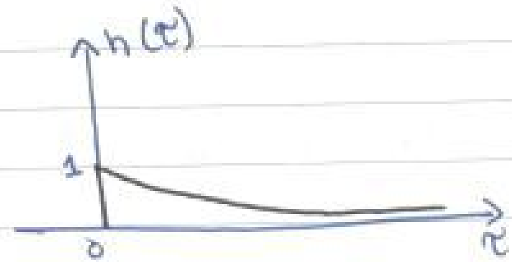
Compute $y(t)$:-

Sol :-

$x(t) = u(t)$, $h(t) = e^{-at} u(t)$, $a > 0$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau$$





For $t<0$, $h(t-\tau)$ and $x(\tau)$ doesnot overlap, while for $t>0$ they overlap from $\tau=0$ to $\tau=t$.

Hence for $t<0$ $y(t)=0$ and for $t>0$, we have.

$$y(t) = x(t) * h(t) = \int_0^t e^{-a(t-\tau)}\, d\tau$$

$$= \int_0^t e^{-at} \cdot e^{a\tau}\, d\tau$$

$$= e^{-at} \int_0^t e^{a\tau}\, d\tau$$

$$= e^{-at} \left[\frac{1}{a} \left(e^{a\tau} \right)_0^t \right] = \frac{e^{-at}}{a} \left[e^{at} - e^{0} \right]$$

$$y(t) = \frac{e^{-at}}{a} \left[e^{at} - 1 \right]$$

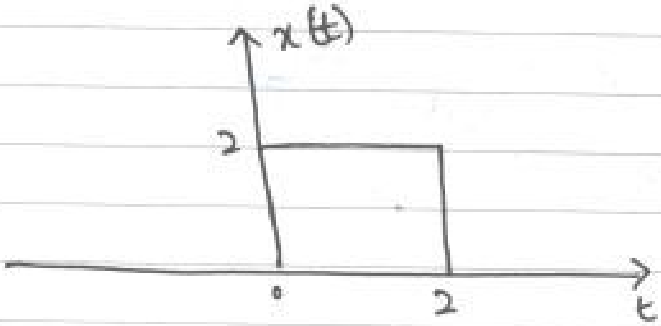
$$= \frac{1}{a} \left[e^{-at+at} - e^{-at} \right] \Rightarrow \frac{1}{a} \left[1 - e^{-at} \right]$$

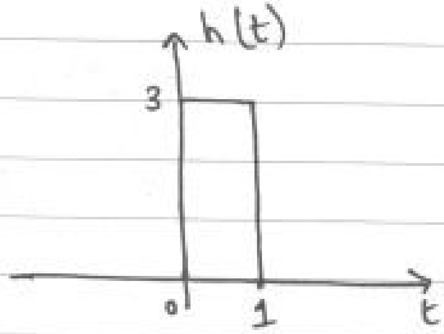
Thus, we can write the output $y(t)$ as

$$y(t) = \frac{1}{a} \left[1 - e^{-at} \right] u(t)$$

EXAMPLE #2:-

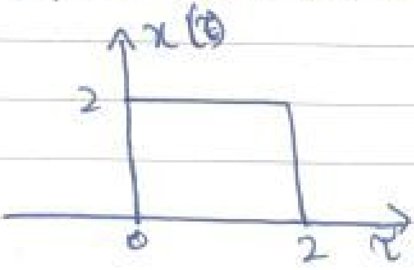
$$y(t) = x(t) * h(t) = ?$$

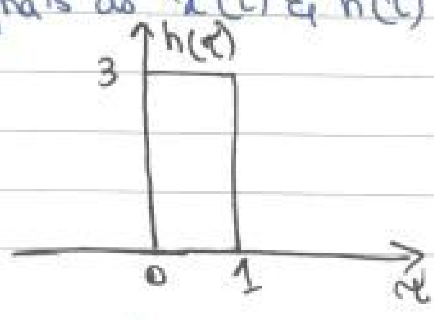




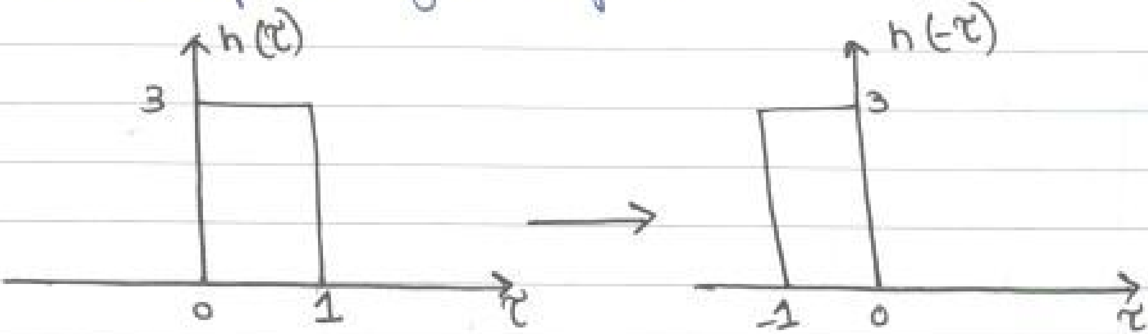
Sol:

Step #1: Re-write the signals as $x(\tau)$ & $h(\tau)$

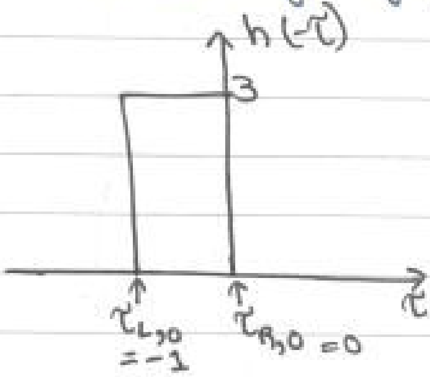




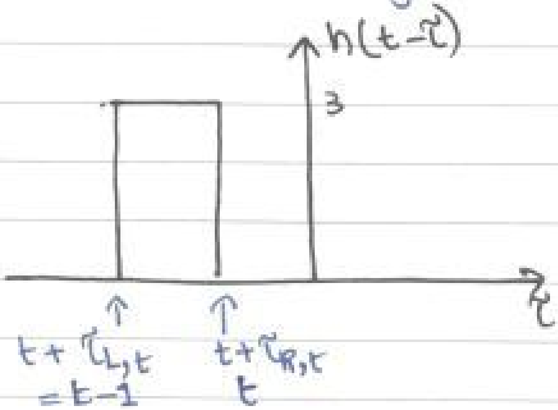
Step #2: Flip one of the signal.



Step #3: Find the edges of fliped signal.



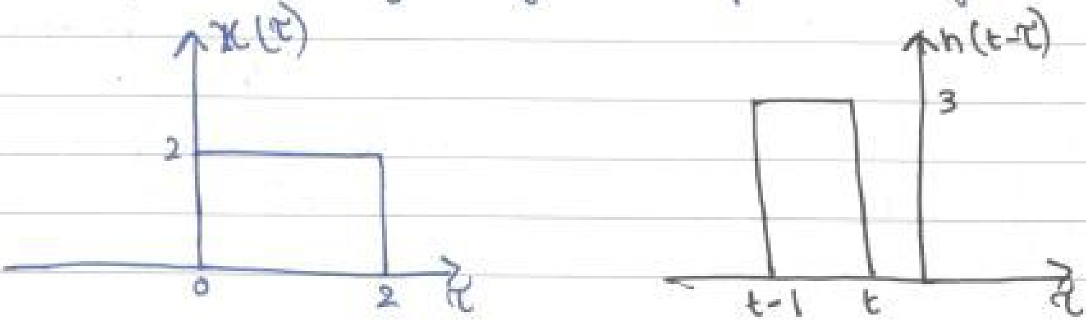
Step #4: Shift $h(-\tau)$ by an arbitrary value of t and get its edges.



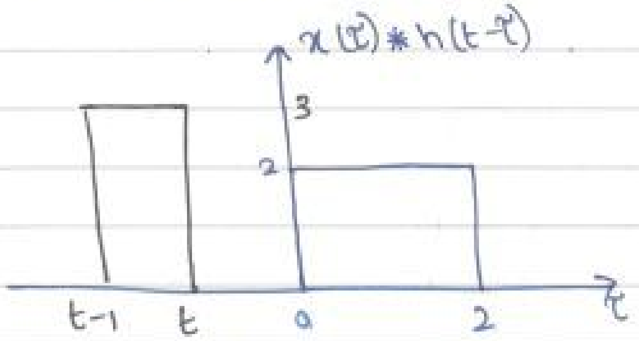
$\tau_{L,0} \Rightarrow t+\tau_{L,t}$

$\tau_{R,0} \Rightarrow t+\tau_{R,t}$

Step #5 & 6: Find regions of τ-overlap and integrate.



Interval 1: when $t < 0$

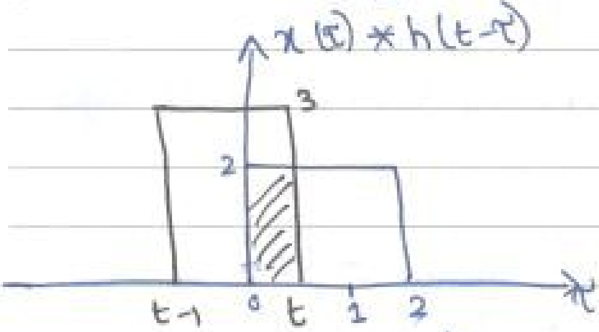


$y(t) = x(t) * h(t) = \{ \; 0$

As there is no overlapping.

⇒ Now start moving signal $h(t-\tau)$ over $x(\tau)$ until it stops overlapping.

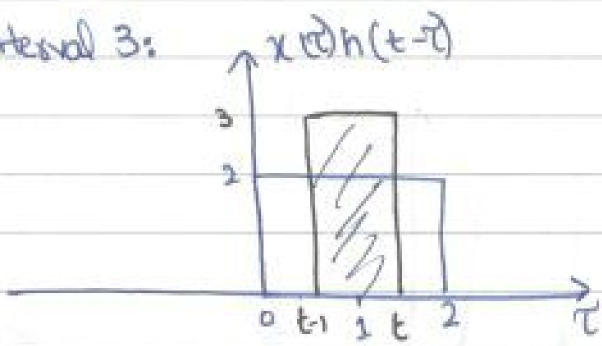
Interval 2: $0 < t \le 1$



$y(t) = x(t) * h(t) = \int_{0}^{t} 6\,d\tau$

$y(t) = 6[\tau]_{0}^{t} = 6[t-0] \Rightarrow 6t$

Interval 3:



$1 < t \le 2$

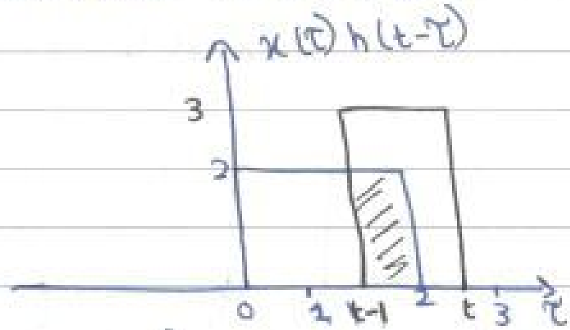
$y(t) = x(t) * h(t) = \int_{t-1}^{t} (2 \times 3)\,d\tau$

$= \int_{t-1}^{t} 6\,d\tau = 6[\tau]_{t-1}^{t}$

$= 6[t-(t-1)] = 6[\cancel{t} - \cancel{t} + 1]$

$y(t) \Rightarrow 6$

Interval 4: $2 < t \le 3$

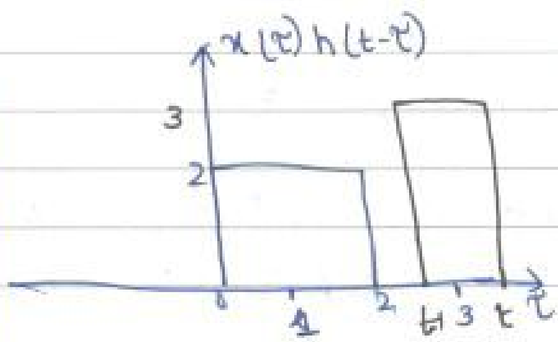


$y(t) = \int_{t-1}^{2} 6\,d\tau = 6[\tau]_{t-1}^{2}$

$= 6[2-(t-1)] = 6[2-t+1]$

$y(t) = 6[3-t] \Rightarrow 18 - 6t$

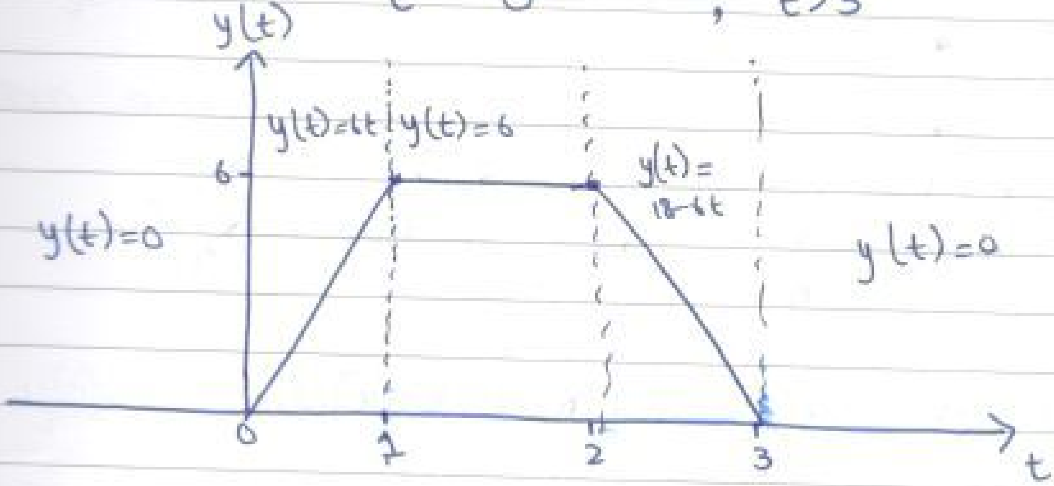
Interval 5: $t > 3$



Hence, no overlapping so $y(t) = 0$

STEP#7: Assemble the output signal.

$$y(t) = \begin{cases} 0 & , t < 0 \\ 6t & , 0 < t \le 1 \\ 6 & , 1 < t \le 2 \\ 18-6t & , 2 < t \le 3 \\ 0 & , t > 3 \end{cases}$$



EXAMPLE#3:

$$x[n] = \alpha^n u[n], \quad h[n] = \beta^n u[n]$$
$$y[n] = x[n] * h[n]$$

Sol:

$$x[n] = \alpha^n u[n], \quad h[n] = \beta^n u[n]$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

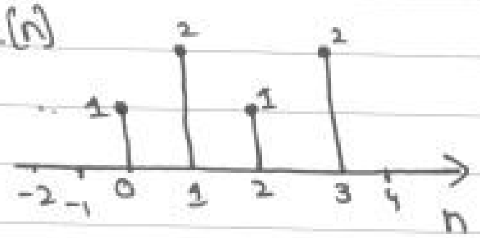
$$= \sum_{k=-\infty}^{\infty} \alpha^k u[k] \beta^{n-k} u[n-k]$$ since $u[k]u[n-k] = \begin{cases} 1 & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$

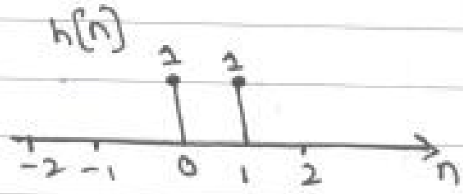
$$= \sum_{k=0}^{n} \alpha^k \beta^{n-k}$$

$$= \sum_{k=0}^{n} \alpha^k \beta^n \beta^{-k} = \beta^n \sum_{k=0}^{n} \left[\frac{\alpha}{\beta}\right]^k \quad n \ge 0$$

$$y[n] = \begin{cases} \beta^n \dfrac{1-(\alpha/\beta)^{n+1}}{1-(\alpha/\beta)} u[n] & \alpha \neq \beta \\ \beta^n (n+1) u[n] & \alpha = \beta \end{cases}$$

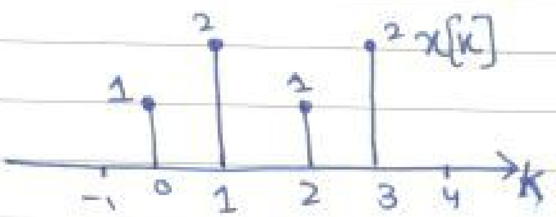
EXAMPLE#4:-

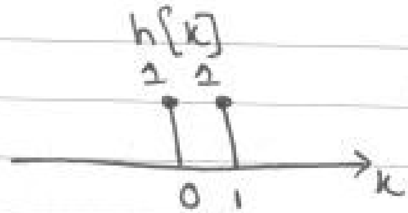




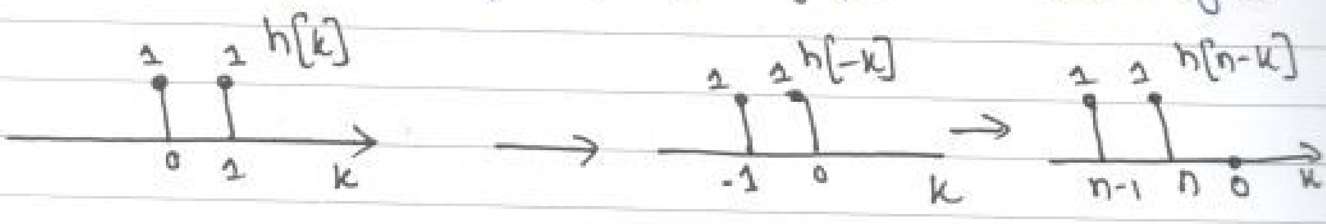
Sol:-

STEP#1:- Re-write the signals as $x[k]$ and $h[k]$.





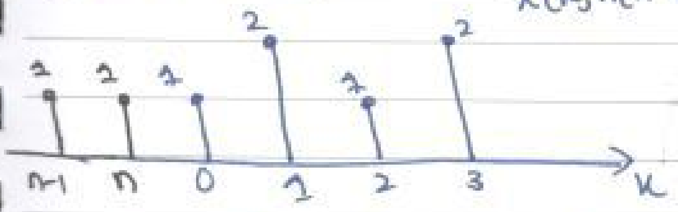
STEP# 2 & 3:- Flip and shift $h[k]$ for any particular value of n.



STEP # 4 & 5:- Slide the signal $h[n-k]$ over $x[k]$ and multiply each overlapping portion and then sum up.

⇒ when $n < 0$.

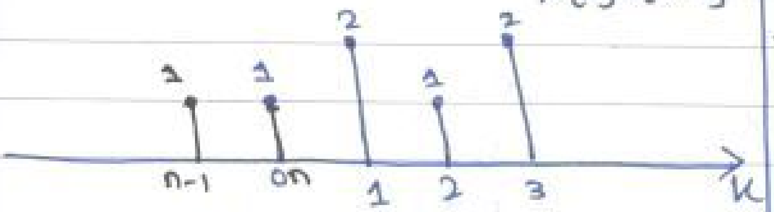
$x[k]h[n-k]$



Hence there is no overlapping
$y[n] = 0$.

⇒ when $n = 0$

$x[k]h[n-k]$

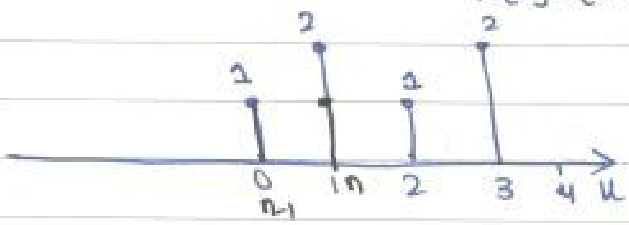


$$y[0] = \sum_{k=-\infty}^{\infty} x[k]\,h[0-k]$$

$$= (1 \times 0) + (1 \times 1) \Rightarrow 1$$

⇒ when $n = 1$

$x[k]h[n-k]$

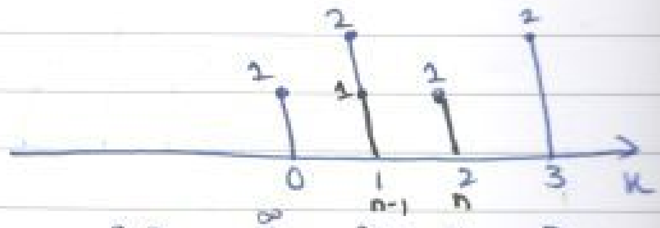


$$y[1] = \sum_{k=-\infty}^{\infty} x[k]\,h[1-k]$$

$$= (1\times1) + (1\times2) \Rightarrow 1+2 \Rightarrow 3$$

⇒ when $n = 2$

$x[k]h[n-k]$

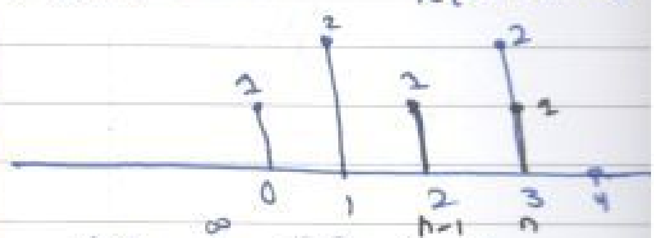


$$y[2] = \sum_{k=-\infty}^{\infty} x[k]\,h[2-k]$$

$$= (1\times2) + (1\times1) \Rightarrow 3$$

⇒ when $n = 3$.

$x[k]h[n-k]$



$$y[3] = \sum_{k=-\infty}^{\infty} x[k]\,h[3-k]$$

$$= (1\times0) + (2\times0) + (1\times1) + (2\times1) \Rightarrow 3$$

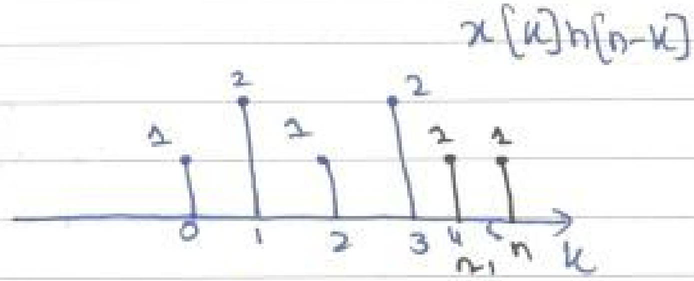
⇒ when $n = 4$

$x[k]h[n-k]$

$$y[4] = \sum_{k=-\infty}^{\infty} x[k]\,h[4-k]$$

$$= (2\times1) + (0\times1) \Rightarrow 2$$

⇒ when $n>4$



Hence, there is no overlapping $y[n]=0$.

<u>STEP#6:-</u> Compile the output signal.

$$y[n] = \begin{cases} 0 & , n<0 \\ 1 & , n=0 \\ 3 & , n=1 \\ 3 & , n=2 \\ 3 & , n=3 \\ 2 & , n=4 \\ 0 & , n>4 \end{cases}$$

$$y[n] = y[0] + y[1] + y[2] + y[3] + y[4]$$

$$= 1 + 3 + 3 + 3 + 2 \Rightarrow 12.$$