Signal & Systems

Lecture # 7

EigenFunctions of an LTI Systems

19th November 18

LTI Systems Described by Differential and Difference Equations

Linear Constant-Coefficient Differential Equations (cont.)

- A general Nth-order linear constant-coefficient differential equation is given by:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- Where coefficients a_k and b_k are real constants.
- The order N refers to the highest derivative of y(t).
- The general solution of above equation for a particular input x(t) is given by:

$$y(t) = y_p(t) + y_h(t)$$

Linear Constant-Coefficient Differential Equations (cont.)

- Where $y_p(t)$ is a particular solution.
- $y_h(t)$ is a homogeneous solution, satisfying the homogeneous differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y_h(t)}{dt^k} = 0$$

Causality of LCCDE

- For the linear system to be causal we must assume the condition of initial rest, i.e., if x(t)=0 for $t \le t_o$, then assume y(t)=0 for $t \le t_o$.
- Thus, the response for $t > t_o$ can be calculated with the initial conditions:

$$y(t_0) = \frac{dy(t_0)}{dt} = \ldots = \frac{d^{N-1}y(t_0)}{dt^{N-1}} = 0$$

where

$$\frac{d^k y(t_0)}{dt^k} = \left.\frac{d^k y(t)}{dt^k}\right|_{t=t_0}$$

Linear Constant-Coefficient Difference Equations

- The Nth-order linear constant coefficient difference equation is: $$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$
- The solution y[n] can be written as the sum of a particular solution and a solution to the homogeneous equation is:

$$\sum_{k=0}^{N} a_k y[n-k] = 0$$

- The solution to this homogeneous equations are often referred to as the natural responses of the system.

Linear Constant-Coefficient Difference Equations (cont.)

- For auxiliary conditions we will focus on the condition of initial rest.
- That is if x[n]=0 for n < n_0, then y[n]=0 for n < n_0.
- With initial rest the system is LTI and causal.
- The above equation can be rearranged in the form:

$$y[n]=\frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k]-\sum_{k=1}^{N} a_k y[n-k]\right\}$$

- These equations are known as recursive equation.
- In the special case when N=0, the above equation reduces to:

$$y[n]=\sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)x[n-k]$$

Linear Constant-Coefficient Difference Equations (cont.)

- Here y[n] is an explicit function of the present and previous values of the input.
- Above equation is also known as non-recursive equation.
- The above equation describes an LTI system and by direct computation, the impulse response of this system is found to be:

$$h[n] = \begin{cases} \dfrac{b_n}{a_0}, & 0 \le n \le M \\ 0, & otherwise \end{cases}$$

- The above equation is nothing more than the convolution sum.

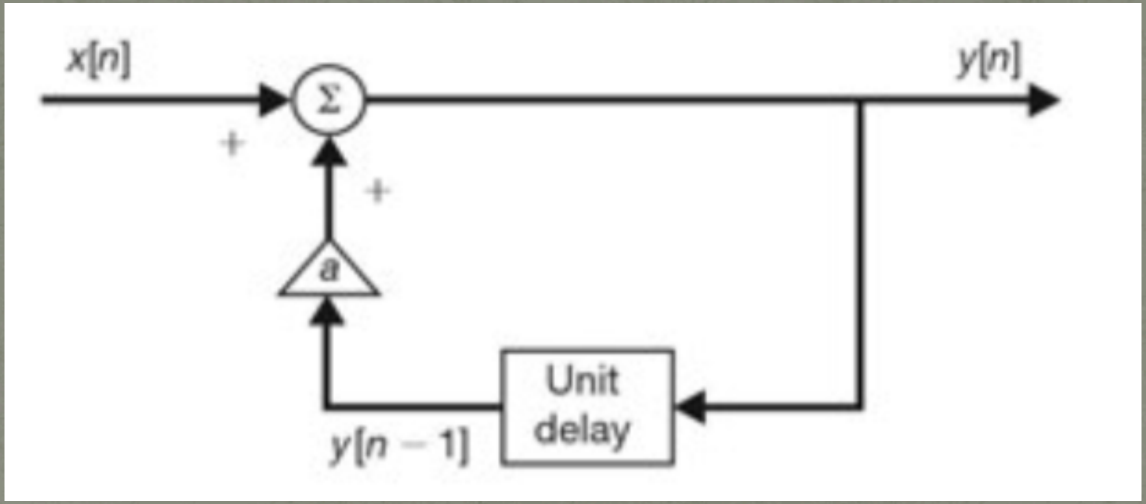
Example #1

- Consider the difference equation:

$$y[n]-\frac{1}{2}y[n-1]=x[n]$$

Example #2

- The discrete-time system shown below consists of one unit delay element and one scalar multiplier. Write a difference equation that relates the output y[n] and the input x[n].



Example #3

- Find the impulse response h[n] for the causal LTI discrete-time system satisfying the following difference equation:

$$y[n]-\frac{1}{2}y[n-2]=2x[n]-x[n-2]$$

Fourier Series

History

- In 1822, the French mathematician J.B.J. Fourier had first studied the periodic function and published his famous theorem.
- It states that any periodic signal can be resolved into sinusoidal components.
- Fourier series is the resulting summation of harmonic sinusoid.
- The signal can be in time domain or in frequency domain.
- T can be represented either in the form of infinite trigonometric series or in the form of exponential series.

Response of LTI Systems

Introduction

- Based on superposition property of LTI systems, response to any input including linear combination of basic signal is the same linear combination of the individual responses to each of the basic signals.
- Continuous-time and Discrete-time periodic signals are described by Fourier Series.
- Aperiodic signals are described by Fourier Transform.

Response of LTI Systems to Complex Exponentials

- For analyzing LTI systems, the signals can be represented as a linear combination of basic signals.
- Basic signals possess the following two properties:
  - The set of basic signals can be used to construct a broad and useful class of signals.
  - Should have simple structure in LTI system response.
- Both of these properties are provided by the set of complex exponential signals in continuous and discrete time.
- The response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude.

Response of LTI Systems to Complex Exponentials (cont.)

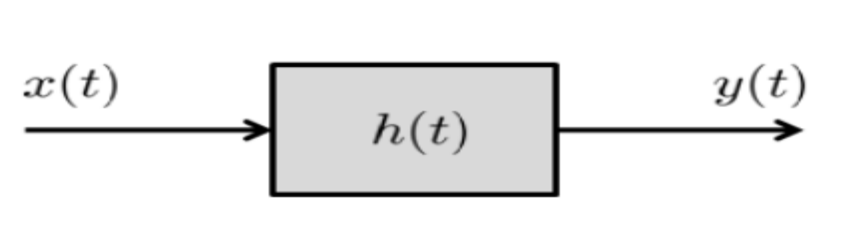
- For Continuous time: $e^{st} \rightarrow H(s)e^{st}$ where H(s) is a function of s.
- For Discrete time: $z^{n} \longrightarrow H(z)z^{n}$ where H(z) is a function of z.

EigenFunction of an LTI Systems

- If the output is a scaled version of its input, then the input function is called an Eigenfunction of the system.
- The scaling factor is called the eigenvalue of the system.

Continuous Time

- Consider an LTI system with impulse response h(t) and input signal x(t).



- Suppose that x(t) = e^{st} for some s belongs to C, then the output is given by:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$

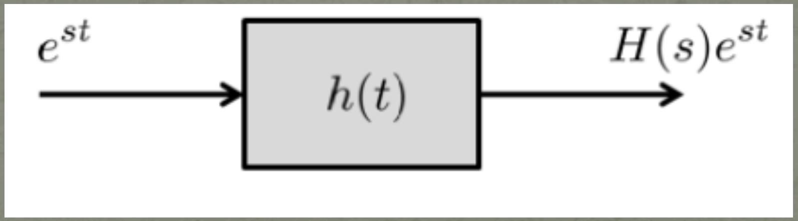
$$= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau$$

Continuous Time (cont.)

$$= e^{st}\left[\int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau\right] = H(s)e^{st} = H(s)x(t)$$

- Where H(s) is defined as: $H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau$

- From the above derivation we see that if the input is $x(t) = e^{st}$, then the output is a scaled version $y(t) = H(s)\, e^{st}$.



Continuous Time (cont.)

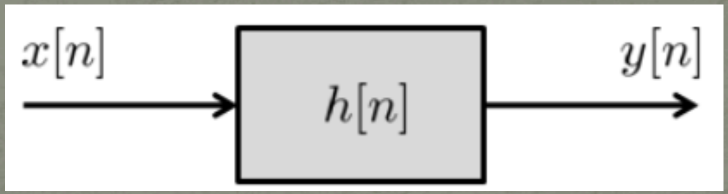
- Therefore, using the definition of Eigen-function, we show that:
  - e^{st} is an Eigenfunction of any continuous-time LTI system
  - H(s) is the corresponding eigenvalue.
- Considering the subclass of periodic complex exponentials of the $e^{j\omega t}$, ω belongs to R by setting s=jω, then:

$$H(s)\big|_{s=j\omega} = H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,d\tau$$

- H(jω) is called the frequency response of the system.

Discrete Time Case

- In parallel manner we can show that complex exponential sequences are Eigenfunctions of discrete-time LTI systems.



- Suppose tat the impulse response is given by h[n] and the input is $x[n]=z^n$, then the output y[n] is:

$$y[n]=h[n]*x[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]$$

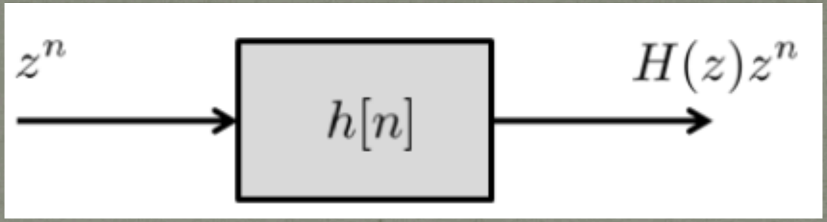
$$=\sum_{k=-\infty}^{\infty}h[k]z^{[n-k]}=z^n\sum_{k=-\infty}^{\infty}h[k]z^{-k}=H(z)z^n$$

- Where:

$$H(z)=\sum_{k=-\infty}^{\infty}h[k]z^{-k}$$

Discrete Time Case (cont.)

- This result indicates:
  - z^n is an Eigenfunction of a discrete-time LTI system
  - $H(z)$ is the corresponding eigenvalue.



- Considering the subclass of periodic complex exponentials $e^{-j(2\pi/N)n}$ by setting $z = e^{j2\pi/N}$, we have:

$$H(z)\Big|_{z=e^{j\Omega}} = H\left(e^{j\Omega}\right) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k}$$

$$where \quad \Omega = \frac{2\pi}{N}$$

Discrete Time Case (cont.)

- And $H(e^{j\Omega})$ is called the frequency response of the system.

Importance of EigenFunction

- The usefulness of Eigenfunctions can be seen from an example.
- Lets consider a signal x(t):

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$$

- According to the Eigenfunction analysis , the output of each complex exponential is:

$$e^{s_1 t} \rightarrow H(s_1) e^{s_1 t}$$

$$e^{s_2 t} \rightarrow H(s_2) e^{s_2 t}$$

$$e^{s_3 t} \rightarrow H(s_3) e^{s_3 t}$$

Importance of EigenFunction (cont.)

- From the superposition property the response to the sum is the sum of the responses, so that:

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$$

- The result implies that if the input is a linear combination of complex exponentials, the output of an LTI system is an infinite sum of complex exponentials.
- More generally, if x(t) is an infinite sum of complex exponentials,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{s_k t}$$

Importance of EigenFunction (cont.)

- Then the output is:

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(s_k) e^{s_k t}$$

- Similarly for discrete-time signals, if:

$$x[n] = \sum_{k=-\infty}^{\infty} a_k z_k^n$$

$$then$$

$$y[n] = \sum_{k=-\infty}^{\infty} a_k H(z_k) z_k^n$$

Importance of EigenFunction (cont.)

- This is an important observation, because as long as we can express a signal x(t) as a linear combination of Eigenfunctions, then the output y(t) can be easily determined by looking at the transfer function. Same goes for discrete-time.
- The transfer function is fixed for an LTI system.

Example #4

- Consider a continuous-time LTI system with the input-output relation given by:

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau)\, d\tau$$

- (a): Find the impulse response h(t) of this system.
- (b): Show that the complex exponential function e^{st} is an Eigenfunction of the system.
- (c): Find the eigenvalue of the system corresponding to e^{st} by using the impulse response h(t) obtained in part (a).

The End