Signal & Systems

Lecture # 8

Fourier Series - 1

26th November 18

Fourier Series

Introduction to Fourier Series

- Fourier series expansion is used for periodic signals to expand them in term of their harmonics which are sinusoidal and orthogonal to one another.
- Fourier series is used for analysis purpose.
- For aperiodic signals we have Fourier transform.

Fourier Series of Continuous-Time Periodic Signals

Fourier Series of Continuous-Time

- According to the definition of periodic signals: x(t) = x(t+T) with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$.
- We have also discussed two basic signals, the sinusoidal signal: $x(t)=\cos\omega_0 t$ and the periodic complex exponential $x(t) = e^{j\omega_0 t}$.
- Both of these signals are periodic with fundamental frequency ω_0 and the fundamental period $T=2\pi/\omega_0$.
- Harmonically related complex exponentials:

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t}, k = 0, \pm 1, \pm 2, \ldots\ldots$$

Fourier Series of Continuous-Time (cont.)

- Each harmonic has fundamental frequency which is multiple of ωo.
- A Linear combination of harmonically related complex exponentials:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

- Above equation is also periodic with period T.
- k=±1 have fundamental frequency ω_o (first harmonic)
- k=±N have fundamental frequency $N\omega_o$ (Nth harmonic)

Continuous-Time Fourier Series Coefficients

- Theorem: The continuous-time Fourier series coefficients a_k of the signal:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad \textit{Synthesis} \quad \textit{Equation}$$

- Is given by:

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} \, dt, \quad \textit{Analysis} \quad \textit{Equation}$$

- Proof:
- Let us consider the signal:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- If we multiply $e^{-jn\omega_0 t}$ on both sides, then we have:

Continuous-Time Fourier Series Coefficients (cont.)

$$x(t)e^{-jn\omega_0 t} = \left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right] e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}$$

- Integrating both sides from 0 to T yields: (T is the fundamental period of x(t))

$$\int_0^T x(t)e^{-jn\omega_0 t}\,dt = \int_0^T \left[\sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}\right] dt$$

$$= \sum_{k=-\infty}^{\infty} \left[a_k \int_0^T e^{j(k-n)\omega_0 t}\,dt\right]$$

Continuous-Time Fourier Series Coefficients (cont.)

- Use Euler's formula:

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt = \int_0^T \cos\left((k-n)\omega_0 t\right)dt + j\int_0^T \sin\left((k-n)\omega_0 t\right)dt$$

- For $k \neq n$, $\cos(k-n)\omega_o t$ and $\sin(k-n)\omega_o t$ are periodic sinusoids with fundamental period ($T/|k-n|$).
- Therefore: $\frac{1}{T}\int_0^T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} 1 & \text{if} \quad k = n \\ 0 & \text{otherwise} \end{cases}$
- This result is known as the orthogonality of complex exponentials.

Continuous-Time Fourier Series Coefficients (cont.)

- Using above equation we have:

$$\int_{0}^{T} x(t)e^{-jn\omega_0 t}\,dt = Ta_n$$

- Which is equivalent to:

$$a_n = \frac{1}{T}\int_{0}^{T} x(t)e^{-jn\omega_0 t}\,dt$$

- Dc or constant component of x(t):

$$a_0 = \frac{1}{T}\int_{T} x(t)\,dt$$

Example #1

- Consider the signal: $x(t) = 1 + \frac{1}{2}\cos 2\pi t + \sin 3\pi t$
- The period of x(t) is T=2, so the fundamental frequency is $\omega_0 = 2\pi/T = \pi$.
- Recall Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$, we have:

$$x(t) = 1 + \frac{1}{4}\left[e^{j2\pi t} + e^{-j2\pi t}\right] + \frac{1}{2j}\left[e^{j3\pi t} - e^{-j3\pi t}\right]$$

$$a_0 = 1, \quad a_1 = a_{-1} = 0, \quad a_2 = a_{-2} = \frac{1}{4}, \quad a_3 = \frac{1}{2j}, \quad a_{-3} = -\frac{1}{2j}$$

$$\textit{and} \quad a_k = 0 \quad \textit{otherwise}$$

Convergence of the Fourier Series

Existence of Fourier Series

- To understand the validity of Fourier Series representation, lets examine the problem of approximation a given periodic signal x(t) by a linear combination of a finite number of harmonically related complex exponentials.
- That is by finite series of the form:

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

- Let $e_N(t)$ denote the approximation error; i.e.,

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

Existence of Fourier Series (cont.)

- The criterion that we will use is the energy in the error over one period:

$$E_N(t) = \int_T |e_N(t)|^2 \, dt$$

- To achieve min E_N, one should define:

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} \, dt$$

- As N increases, E_N decreases and as $N \rightarrow \infty$ E_N is zero.
- If $a_k \rightarrow \infty$ the approximation will diverge.
- Even for bounded a_k the approximation may not be applicable for all periodic signals.

Convergence Conditions of Fourier Series Approximation

- Energy of signal should be a finite in a period:

$$\int_T |x(t)|^2 \, dt < \infty$$

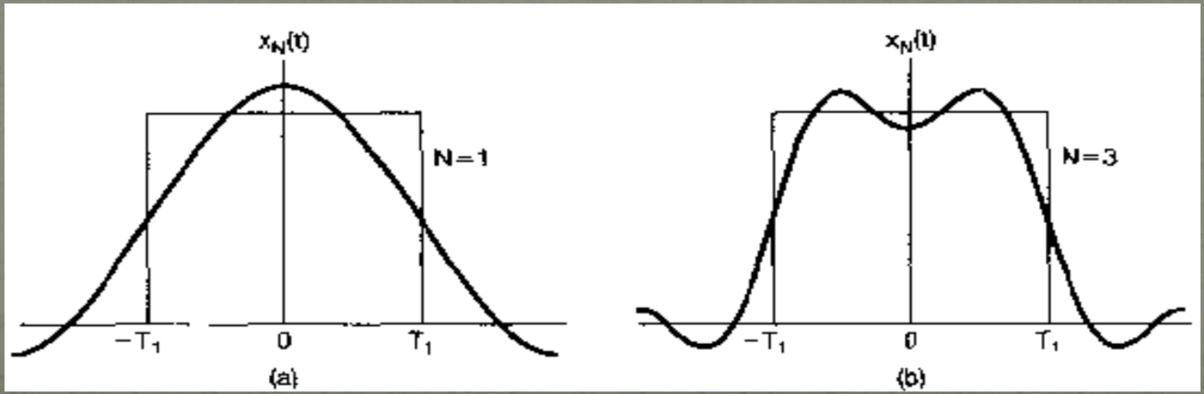
  - This condition only guarantees EN\o.
  - It does not guarantee that x(t) equals to its Fourier series at each moment t.

Convergence Conditions of Fourier Series Approximation (cont.)

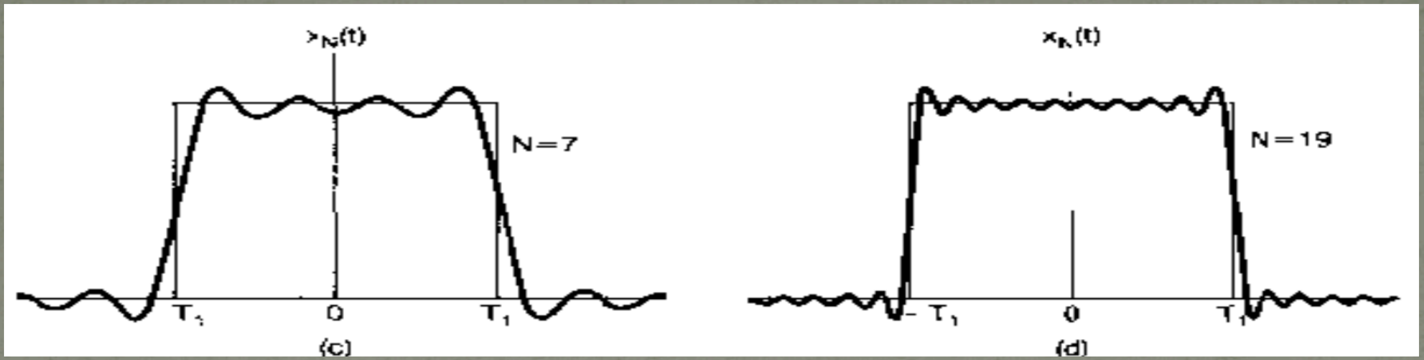
- Dirichlet Conditions:
  - Over any period x(t) must be absolutely integrable.
  - In any finite interval of time x(t) is of bounded variation, i.e., there are no more than a finite number of maxima and minima during any single period of the signal.
  - In any finite interval of time, there are only a finite number of discontinuities.

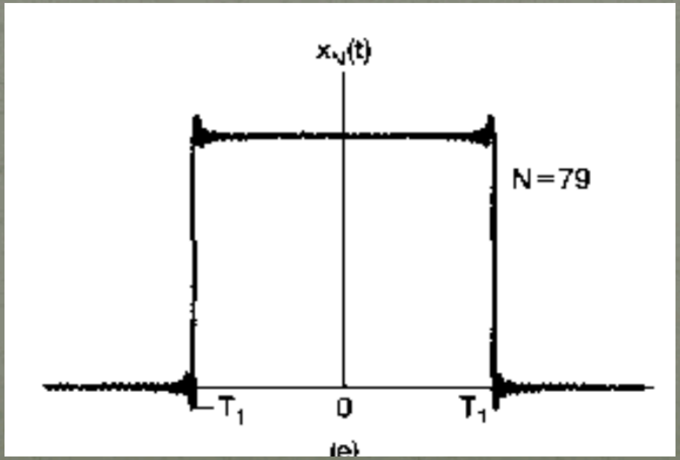
Gibbs Phenomenon

- Near a point where x(t) has a jump discontinuity, the partial sums x_N (t) of a Fourier series exhibit a substantial overshoot near these endpoints.
- An increase in N will not diminish the amplitude of the overshoot, although with increasing N the overshoot occurs over smaller and smaller intervals.
- This phenomenon is known as Gibbs Phenomenon.



Gibbs Phenomenon (cont.)





Fourier Series Representation of Discrete-Time Periodic Signals

Fourier Series Representation of Discrete Time

- The Fourier series representation of a discrete-time periodic signal is finite as opposed to the infinite series representation required for continuous-time periodic signals.

Linear Combinations of Harmonically Related Complex Exponentials

- A discrete-time signal x[n] is periodic with period N if: x[n] = x[n+N].
- The fundamental period is the smallest positive N and the fundamental frequency is $\omega_0 = \frac{2\pi}{N}$.
- The set of all discrete-time complex exponential signals that are periodic with period N is given by:

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk(2\pi/N)n}, \quad k = 0, \pm 1, \pm 2, \ldots..$$

- All of these signals have fundamental frequencies that are multiples of 2π/N and thus are harmonically related.

Linear Combinations of Harmonically Related Complex Exponentials (cont.)

- There are only N distinct signals in the set this is because the discrete-time complex exponentials which differ in frequency by a multiple of 2π are identical. That is:

$$\phi_k[n] = \phi_{k+rN}[n]$$

- The representation of periodic sequences in terms of linear combinations of the sequences $\Phi_k[n]$ is:

$$x[n] = \sum_k a_k \phi_k[n] = \sum_k a_k e^{jk\omega_0 n} = \sum_k a_k e^{jk(2\pi/N)n}$$

- Since the sequences $\Phi_k[n]$ are distinct over a range of N successive values of k, the summation in above equation need include terms over this range.

Linear Combinations of Harmonically Related Complex Exponentials (cont.)

- Thus the summation is on k as k varies over a range of N successive integers beginning with any value of k.
- We indicate this by expressing the limits of the summation as k=<N>. That is:

$$x[n] = \sum_{k=\langle N \rangle} a_k \phi_k[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

Discrete-Time Fourier Series Coefficients

- Assuming x[n] is square-summable i.e., $\sum_{n=-\infty}^{\infty} \left| x[n] \right|^2 < \infty$ or x[n] satisfies the Dirichlet conditions.
- In this case we have:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}, \quad \textit{Synthesis} \quad \textit{Equation}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}, \quad \textit{Analysis} \quad \textit{Equation}$$

- As in continuous time, the discrete-time Fourier series coefficient a_k are often referred to as the spectral coefficients of x[n].

Discrete-Time Fourier Series Coefficients (cont.)

- These coefficients specify a decomposition of x[n] into a sum of N harmonically related complex exponentials.

Example #2

- Consider the signal:

$$x[n] = \sin \omega_0 n$$

- Which is the discrete-time counterpart of the signal $x(t) = \sin \omega_0 t$.
- x[n] is periodic only if 2π/ωo is an integer or a ratio of integers.

The End