Signal & Systems

Lecture # 10

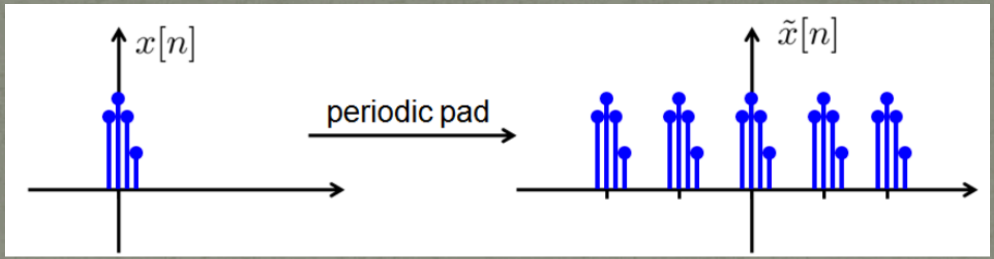
DTFT-I

14th December 18

Discrete Time Fourier Transform

Development of the Discrete-Time Fourier Transform

- In deriving discrete-time Fourier Transform we have three key steps:
- Step#1:
  - Consider an aperiodic discrete-time signal x[n]. We pad x[n] to construct a periodic signal x'[n].



- Step#2:
- Since x'[n] is periodic, by discrete-time Fourier series we have:

$$x'[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

- Where a_k is:

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x'[n] e^{-jk(2\pi/N)n}$$

Development of the Discrete-Time Fourier Transform (cont.)

- Here, $\omega_o=2\pi/N$.
- Now note that x'[n] is a periodic signal with period N and the non-zero entries of x'[n] in a period are the same as the non-zero entries of x[n].
- Therefore, it holds that:

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x'[n]e^{-jk(2\pi/N)n}$$

$$= \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-jk(2\pi/N)n}$$

- If we define:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Development of the Discrete-Time Fourier Transform (cont.)

- Then:

$$a_k = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk(2\pi/N)n} = \frac{1}{N} X\left(e^{jk\omega_0}\right)$$

- Step#3:
  - Putting above equation in discrete-time Fourier series equation, we have:

$$x'[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$$

$$= \sum_{k=\langle N \rangle} \left[\frac{1}{N} X\left(e^{jk\omega_0}\right) \right] e^{jk\omega_0 n}$$

$$= \frac{1}{2\pi} \sum_{k=\langle N \rangle} X\left(e^{jk\omega_0}\right) e^{jk\omega_0 n} \omega_0, \quad \omega_0 = \frac{2\pi}{N}$$

Development of the Discrete-Time Fourier Transform (cont.)

- As N➔∞,ω_0 ➔0, so the area becomes infinitesimal small and sum becomes integration and x'[n]=x[n],so above equation becomes,

$$x'[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X\left(e^{jk\omega_0}\right) e^{jk\omega_0 n} \omega_0 \rightarrow \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} \, d\omega$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} \, d\omega$$

- Hence, the Discrete time Fourier transform pair:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Development of the Discrete-Time Fourier Transform (cont.)

- The first equation is referred to as synthesis equation and second one as analysis equation.
- $X(e^{j\omega})$ is referred to as the spectrum of $x[n]$.

Is $X(e^{j\omega})$ Periodic?

Why is $X(e^{j\omega})$ Periodic?

- The continuous time Fourier transform $X(j\omega)$ is aperiodic in general but the discrete time Fourier transform $X(e^{j\omega})$ is always periodic.
- To prove this, let us consider the discrete-time Fourier transform, here we want to check whether:

$$X\left(e^{j\omega}\right) = X\left(e^{j(\omega+2\pi)}\right)?$$

$$X\left(e^{j(\omega+2\pi)}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\left(e^{-j2\pi}\right)^{n} = X\left(e^{j\omega}\right)$$

Why is $X(e^{j\omega})$ Periodic? (cont.)

- Because $(e^{-j2\pi})^n = 1^n = 1$, for any integer n. Therefore, $X(e^{j\omega})$ is periodic with period 2π.
- Now, let us consider the continuous-time Fourier transform, and check the periodicity for it,

$$X(j\omega) = X(j(\omega + 2\pi))?$$

$$X(j(\omega + 2\pi)) = \int_{-\infty}^{\infty} x(t)e^{-j(\omega+2\pi)t}\,dt = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}(e^{-j2\pi})^t\,dt$$

- Here t is a real number and is from $-\infty$ to ∞. But $e^{-j2\pi t} \neq 1$ unless t is an integer and in the case of discrete time n is always an integer.

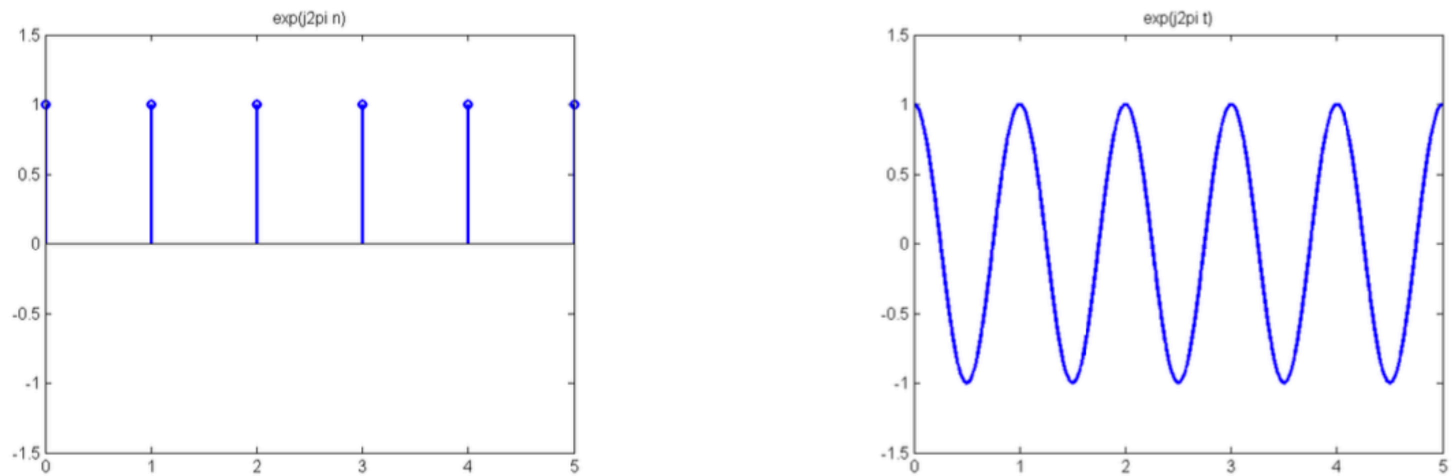
Why is $X(e^{j\omega})$ Periodic?

- Therefore:

$$\int_{-\infty}^{\infty} x(t)e^{-j\omega t}(e^{-j2\pi})^t\,dt \neq \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$X(j(\omega + 2\pi)) \neq X(j\omega)$$

Why is X($e^{j\omega}$) Periodic? (cont.)



(a) $(e^{j2\pi})^n = 1$ for all n, because n is integer.

(b) $(e^{j2\pi})^t \neq 1$ unless t is an integer.

Example #1

- Consider the signal:

$$x[n] = a^n u[n], \quad |a| < 1$$

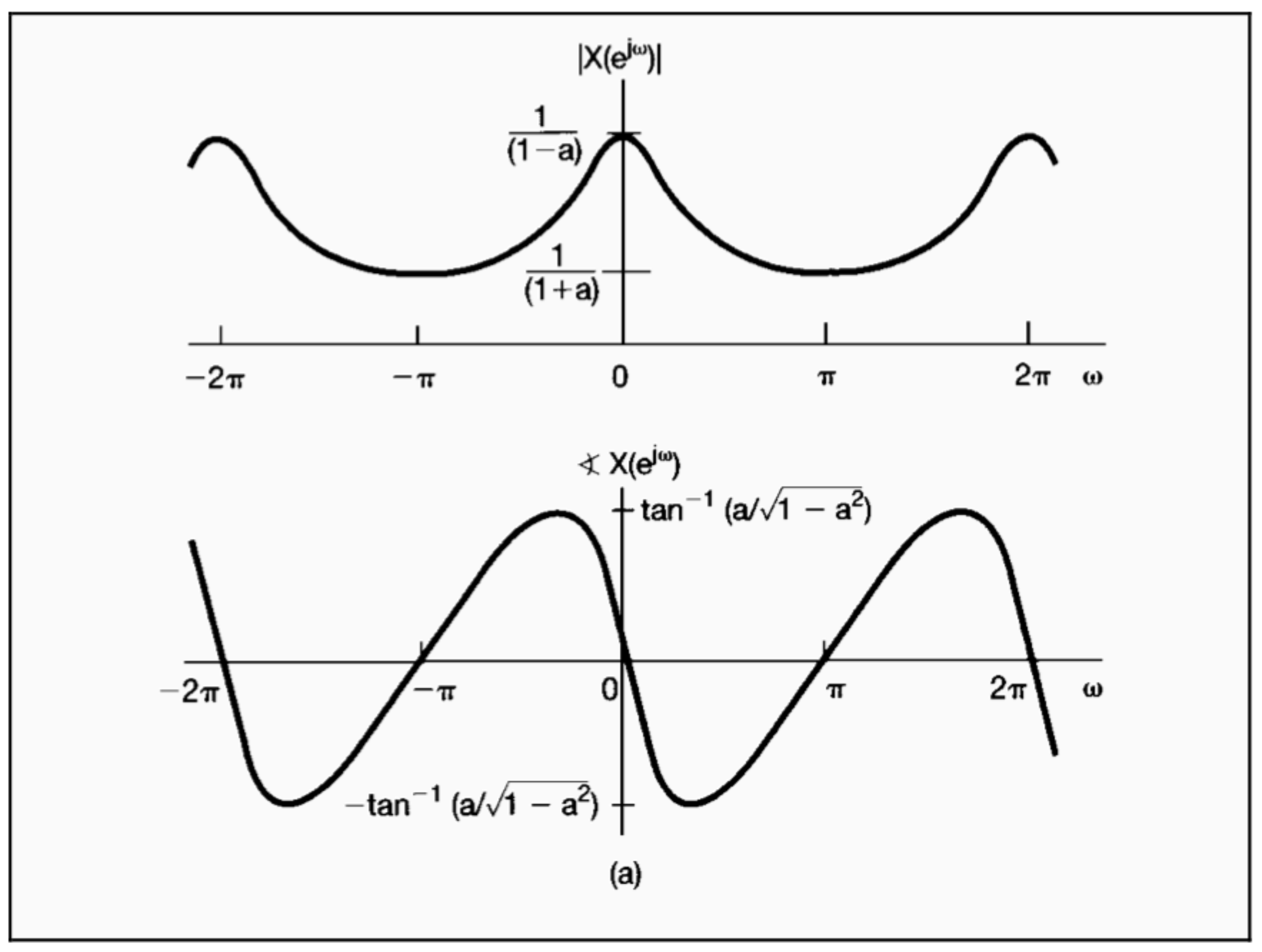
- Solution:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n u[n]e^{-j\omega n}$$

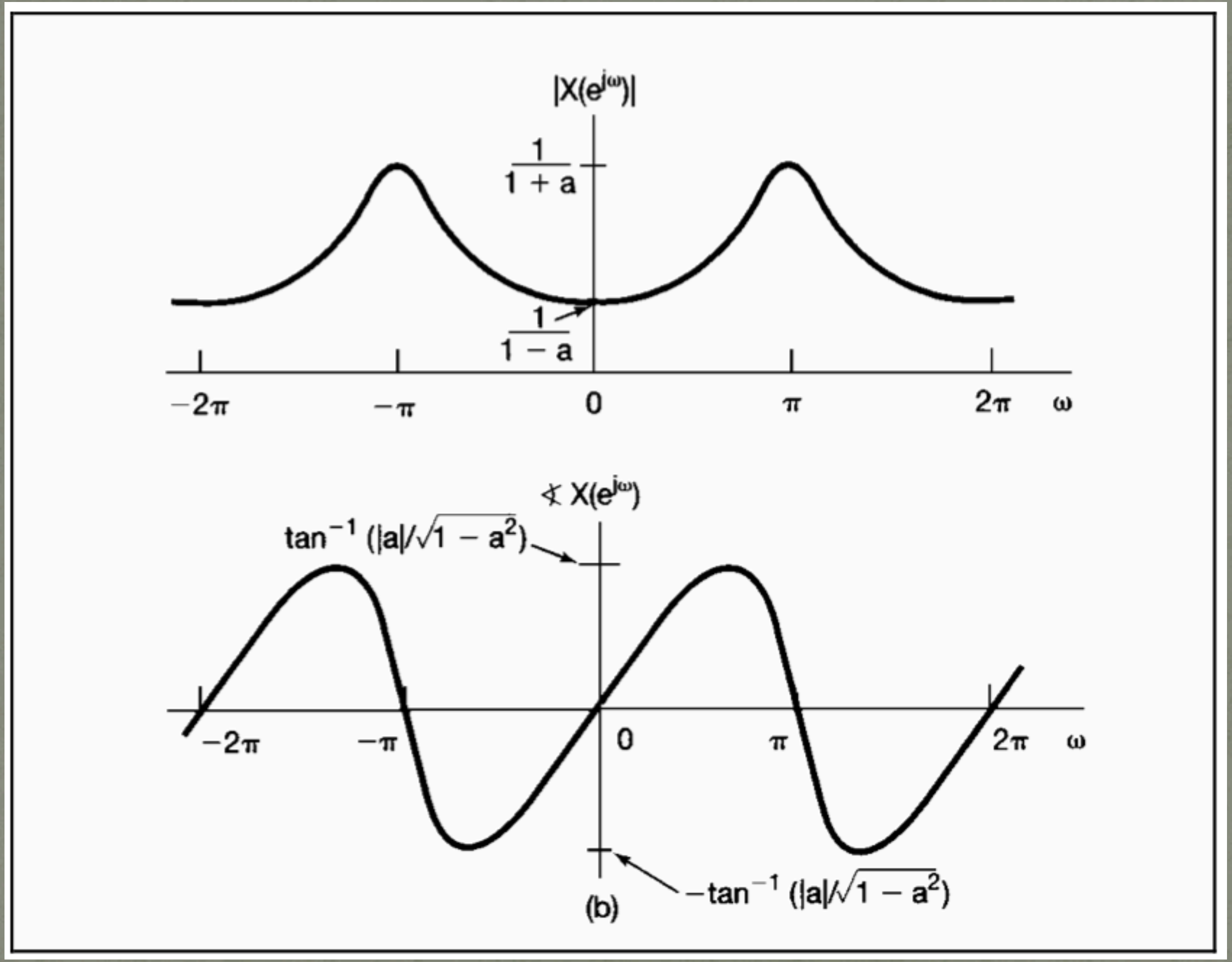
$$= \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n = \frac{1}{1 - ae^{-j\omega}}$$

- The magnitude and phase for this example are shown in the figure below, where a>o and a<o are shown in figure a and b.

Example #1 (cont.)



Example #1 (cont.)



Example #2

- Consider the signal:

$$x[n] = a^{|n|}, \quad |a| < 1$$

- Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} u[n] e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

- Let m=-n in the first summation we obtain,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} a^{n} e^{-j\omega n} + \sum_{m=1}^{\infty} a^{m} e^{j\omega m}$$

$$= \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^{n} + \sum_{m=1}^{\infty} \left(ae^{j\omega}\right)^{m}$$

Example #2 (cont.)

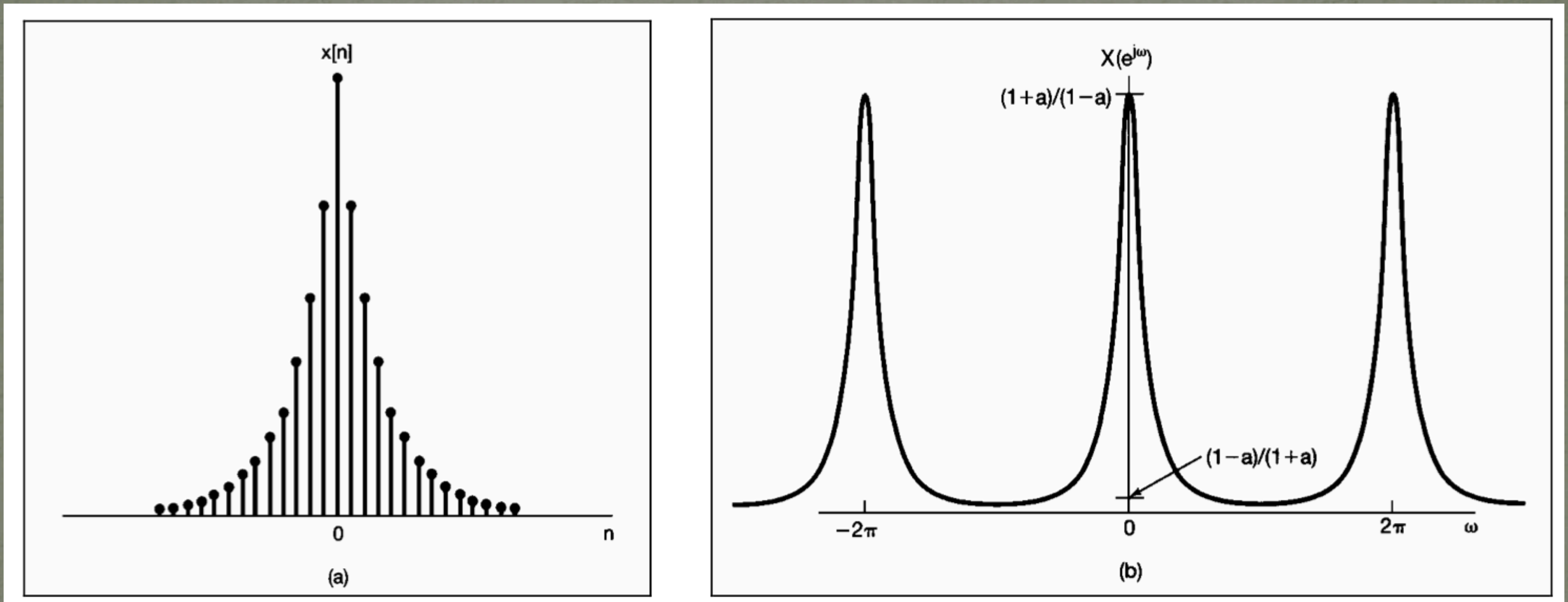
- Both of these summations are infinite geometric series that we can evaluate in closed form, yielding:

$$X\left(e^{j\omega}\right) = \frac{1}{1-ae^{-j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}}$$

$$= \frac{1-a^2}{1-2a\cos\omega + a^2}$$

- Figure (a) below is signal x[n] = a|n| and figure (b) is its Fourier transform (0 < a < 1)

Example #2 (cont.)



Fourier Transform for Periodic Signals

Periodic Signals

- For a periodic discrete-time signal:

$$x[n] = e^{j\omega_0 n}$$

- The discrete-time Fourier transform must be periodic in ω with period 2π.
- Then the Fourier transform of x[n] should have impulses at ω_o , $\omega_o \pm 2\pi$, $\omega_o \pm 4\pi$, and so on.
- In fact, the Fourier transform of x[n] is the impulse train:

$$X\left(e^{j\omega}\right) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

Periodic Signals (cont.)

- Now consider a periodic sequence x[n] with period N and with the Fourier series representation.

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

- In this case, the Fourier transform is:

$$X\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

- So that the Fourier transform of a periodic signal can directly constructed from its Fourier coefficients.

Example #3

- Consider the periodic signal:

$$x[n] = \cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}, \quad where \quad \omega_0 = \frac{2\pi}{5}$$

- Solution:
  - From the equation of periodicity we can write:

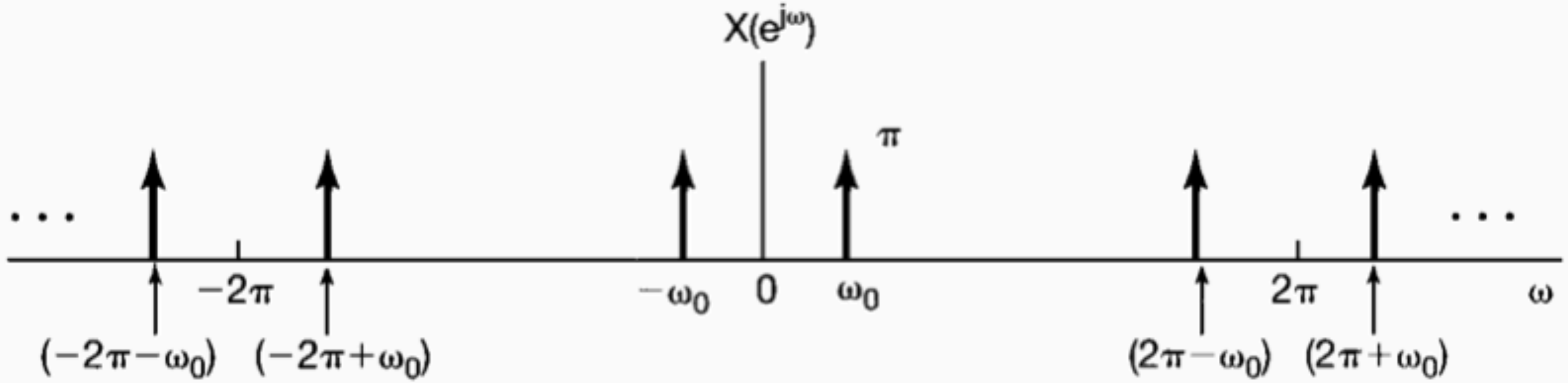
$$X\left(e^{j\omega}\right) = \sum_{l=-\infty}^{\infty} \pi\delta(\omega - \frac{2\pi}{5} - 2\pi l) + \sum_{l=-\infty}^{\infty} \pi\delta(\omega + \frac{2\pi}{5} - 2\pi l)$$

  - That is,

$$X\left(e^{j\omega}\right) = \pi\delta\left(\omega - \frac{2\pi}{5}\right) + \pi\delta\left(\omega + \frac{2\pi}{5}\right), \quad -\pi \le \omega < \pi$$

  - $X(e^{j\omega})$ repeats periodically with a period of 2π, as shown below:

Example #3 (cont.)



Discrete-time Fourier transform of $x[n] = \cos \omega_0 n$.

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