Signal & Systems

Lecture # 11

DTFT-II

21st December 18

Properties of Discrete Time Fourier Transform

Periodicity

- The discrete-time Fourier transform is always periodic in ω with period 2π, i.e.,

$$X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right)$$

Linearity

- If:

$$x_1[n] \leftrightarrow X_1\left(e^{j\omega}\right)$$

$$And$$

$$x_2[n] \leftrightarrow X_2\left(e^{j\omega}\right)$$

- Then:

$$ax_1[n] + bx_2[n] \overset{F}{\leftrightarrow} aX_1\left(e^{j\omega}\right) + bX_2\left(e^{j\omega}\right)$$

Time Shifting & Frequency Shifting

- If:

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

- Then:

$$x[n-n_0] \overset{F}{\leftrightarrow} e^{-j\omega_0 n} X\left(e^{j\omega}\right)$$

$$and$$

$$e^{j\omega_0 n} x[n] \overset{F}{\leftrightarrow} X\left(e^{j(\omega-\omega_0)}\right)$$

Conjugation & Conjugate Symmetry

- If:

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

- Then:

$$x^*[n] \overset{F}{\leftrightarrow} X^*\left(e^{-j\omega}\right)$$

- If x[n] is real valued, its transform $X(e^{j\omega})$ is conjugate symmetric. That is: $X\left(e^{j\omega}\right) = X^*\left(e^{-j\omega}\right)$
- From this, it follows that $\mathrm{Re}\left\{X\left(e^{j\omega}\right)\right\}$ is an even function of ω and $\mathrm{Im}\left\{X\left(e^{j\omega}\right)\right\}$ is an odd function of ω.
- Similarly the magnitude of $X(e^{j\omega})$ is an even function and the phase angle is an odd function.

Conjugation & Conjugate Symmetry (cont.)

- Furthermore,

$$Ev\left\{x[n]\right\} \overset{F}{\leftrightarrow} \mathrm{Re}\left\{X\left(e^{j\omega}\right)\right\}$$

and

$$Od\left\{x[n]\right\} \overset{F}{\leftrightarrow} j\,\mathrm{Im}\left\{X\left(e^{j\omega}\right)\right\}$$

Differencing & Accumulation

- If:

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

- Then:

$$x[n]-x[n-1] \overset{F}{\longleftrightarrow} \left(1-e^{-j\omega}\right)X\left(e^{j\omega}\right)$$

$$For \quad signal,$$

$$y[n] = \sum_{m=-\infty}^{n} x[m],$$

- Its Fourier transform is given as:

$$\sum_{m=-\infty}^{n} x[m] \overset{F}{\longleftrightarrow} \frac{1}{\left(1-e^{-j\omega}\right)} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{m=-\infty}^{+\infty} \delta(\omega - 2\pi k)$$

Time Reversal

- If:

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

- Then:

$$x[-n] \overset{F}{\longleftrightarrow} X\left(-e^{j\omega}\right)$$

Differentiation in Frequency

- If:

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

- Then:

$$nx[n] \overset{F}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

Parseval's Relation

- If:

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

- Then:

$$\sum_{n=-\infty}^{+\infty} \left|x[n]\right|^2 = \frac{1}{2\pi} \int_{2\pi} \left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

Convolution Property

- If x[n], h[n] and y[n] are the input, impulse response, and output respectively, of an LTI system, so that,

$$y[n] = x[n] * h[n]$$

$$then,$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- Where $X(e^{j\omega})$, $H(e^{j\omega})$ and $Y(e^{j\omega})$ are the Fourier transforms of x[n], h[n] and y[n] respectively.

Multiplication Property

- It states that:

$$y[n] = x_1[n]x_2[n] \overset{F}{\longleftrightarrow} Y\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{2\pi} X_1\left(e^{j\omega}\right)X_2\left(e^{j(\omega-\theta)}\right)d\theta$$

Properties of DTFT

PROPERTY	SEQUENCE	FOURIER TRANSFORM
	$x[n]$	$X(\Omega)$
	$x_1[n]$	$X_1(\Omega)$
	$x_2[n]$	$X_2(\Omega)$
Periodicity	$x[n]$	$X(\Omega + 2\pi) = X(\Omega)$
Linearity	$a_1x_1[n] + a_2x_2[n]$	$a_1X_1(\Omega) + a_2X_2(\Omega)$
Time shifting	$x[n - n_0]$	$e^{-j\Omega n0}X(\Omega)$
Frequency shifting	$e^{j\Omega_0 n}x[n]$	$X(\Omega - \Omega_0)$
Conjugation	$x^*[n]$	$X^*(-\Omega)$
Time reversal	$x[-n]$	$X(-\Omega)$
Time scaling	$x_{(m)}[n] = \begin{cases} x[n/m] & \text{if } n = km \\ 0 & \text{if } n \neq km \end{cases}$	$X(m\Omega)$
Frequency differentiation	$nx[n]$	$j\dfrac{dX(\Omega)}{d\Omega}$
First difference	$x[n] - x[n-1]$	$(1 - e^{-j\Omega})X(\Omega)$

Properties of DTFT (cont.)

Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\pi X(0)\delta(\Omega) + \frac{1}{1-e^{-j\Omega}} X(\Omega)$ $\lvert\Omega\rvert \leq \pi$
Convolution	$x_1[n] * x_2[n]$	$X_1(\Omega)X_2(\Omega)$
Multiplication	$x_1[n]x_2[n]$	$\frac{1}{2\pi} X_1(\Omega) \otimes X_2(\Omega)$
Real sequence	$x[n] = x_e[n] + x_o[n]$	$X(\Omega) = A(\Omega) + jB(\Omega)$ $X(-\Omega) = X^*(\Omega)$
Even component	$x_e[n]$	$\mathrm{Re}\{X(\Omega)\} = A(\Omega)$
Odd component	$x_o[n]$	$j\,\mathrm{Im}\{X(\Omega)\} = jB(\Omega)$

Parseval's theorem

$$\sum_{n=-\infty}^{\infty} x_1[n]x_2[n] = \frac{1}{2\pi}\int_{2\pi} X_1(\Omega)X_2(-\Omega)\, d\Omega$$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(\Omega|^2\, d\Omega$$

Common Fourier Transform Pairs

$x[n]$	$X(\Omega)$
$\delta[n]$	1
$\delta(n-n_0)$	$e^{-j\Omega n_0}$
$x[n]=1$	$2\pi\delta(\Omega), \lvert\Omega\rvert \le \pi$
$e^{j\Omega_0 n}$	$2\pi\delta(\Omega-\Omega_0), \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$\cos \Omega_0 n$	$\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)], \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$\sin \Omega_0 n$	$-j\pi[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)], \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$u[n]$	$\pi\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}, \lvert\Omega\rvert \le \pi$
$-u[-n-1]$	$-\pi\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}, \lvert\Omega\rvert \le \pi$
$a^n u[n], \lvert a\rvert < 1$	$\dfrac{1}{1-ae^{-j\Omega}}$
$-a^n u[-n-1], \lvert a\rvert > 1$	$\dfrac{1}{1-ae^{-j\Omega}}$

Common Fourier Transform Pairs (cont.)

	$1 - ae$
$(n+1)\,a^n u[n], \lvert a \rvert < 1$	$\dfrac{1}{\left(1 - ae^{-j\Omega}\right)^2}$
$a^{\lvert n \rvert}, \lvert a \rvert < 1$	$\dfrac{1 - a^2}{1 - 2a\cos\Omega + a^2}$
$x[n] = \begin{cases} 1 & \lvert n \rvert \le N_1 \\ 0 & \lvert n \rvert > N_1 \end{cases}$	$\dfrac{\sin\left[\Omega\left(N_1 + \frac{1}{2}\right)\right]}{\sin(\Omega / 2)}$
$\dfrac{\sin Wn}{\pi n}, 0 < W < \pi$	$X(\Omega) = \begin{cases} 1 & 0 \le \lvert \Omega \rvert \le W \\ 0 & W < \lvert \Omega \rvert \le \pi \end{cases}$
$\sum_{k=-\infty}^{\infty} \delta[n - kN_0]$	$\Omega_0 \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_0), \Omega_0 = \dfrac{2\pi}{N_0}$

Example #1

- Consider the signal:

$$x[n] = \delta[n] + \delta[n-1] + \delta[n+1]$$

- Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\delta[n] + \delta[n-1] + \delta[n+1]\right) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} + \sum_{n=-\infty}^{\infty} \delta[n-1] e^{-j\omega n} + \sum_{n=-\infty}^{\infty} \delta[n+1] e^{-j\omega n}$$

$$X(e^{j\omega}) = 1 + e^{-j\omega} + e^{j\omega} = 1 + 2\cos\omega$$

Example #2

- Given that x[n] has Fourier transform $X(e^{j\omega})$, express the Fourier transforms of the following signals in terms of $X(e^{j\omega})$, by using the Fourier transform properties:

  - (a): $x[n] = x[1-n] + x[-1-n]$
  - (b): $x[n] = (n-1)^2 x[n]$

Systems Characterized by Linear Constant-Coefficient Difference Equations

Linear Constant-Coefficient Difference Equations

- A general linear constant-coefficient difference equation for an LTI system with input x[n] and output y[n] is of the form,

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- Which is usually referred to as Nth-order difference equation.
- If $x[n] = e^{j\omega n}$ is the input to an LTI system, then the output must be of the form $H(e^{j\omega})e^{j\omega n}$. Substituting these expressions into above equation and performing some algebra allow us to solve for $H(e^{j\omega})$.

Linear Constant-Coefficient Difference Equations (cont.)

- Based on convolution, above equation can be written as:

$$H\left(e^{j\omega}\right) = \frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)}$$

- Applying the Fourier transform to both sides and using the linearity and time-shifting properties we obtain the following expression:

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y\left(e^{j\omega}\right) = \sum_{k=0}^{M} b_k e^{-jk\omega} X\left(e^{j\omega}\right)$$

Linear Constant-Coefficient Difference Equations (cont.)

- Or equivalently

$$H\left(e^{j\omega}\right)=\frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)}=\frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

- The frequency response of the LTI system can be written down by inspection as well.

Example #3

- Consider a causal LTI system that is characterized by the difference equation:

$$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n]$$

- And let the input to this system be:

$$x[n]=\left(\frac{1}{4}\right)^{n}u[n]$$

- Find y[n].

The End