# Signal & Systems

Lecture # 12

Z-Transform -I

31st December 18

# Z-Transform

# The Z-Transform

- The z-transform of a discrete-time signal x[n] is:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- The z-transform operation is denoted as:

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

- Where "z" is the complex number. Therefore, we may write z as:

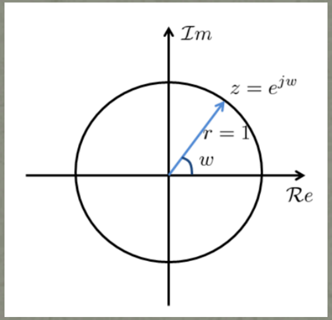
$$z = re^{j\omega}$$

- Where r and ω belongs to Real number. When r=1, the z-transform of a discrete-time signal becomes:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

# The Z-Transform (cont.)

- Therefore, the DTFT is a special case of the z-transform.
- Pictorially, we can view DTFT as the z-transform evaluated on the unit circle:




- When r≠1, the z-transform is equivalent to:

$$X\left(re^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\left(re^{j\omega}\right)^{-n}$$

# The Z-Transform (cont.)

$$= \sum_{n=-\infty}^{\infty} \left( r^{-n} x[n] \right) e^{-j\omega n}$$

$$= F\left[ r^{-n} x[n] \right]$$

- Which is the DTFT of the signal $r^{-n}$ x[n].
- However, we know that the DTFT does not always exist. It exists only when the signal is square summable, or satisfies the Dirichlet conditions.
- Therefore, X(z) does not always converge. It converges only for some values of r. this range of r is called the region of convergence (ROC).

# Region of Convergence (ROC)

- The region of convergence are the values of for which the z-transform converges.
- Z-transform is an infinite power series which is not always convergent for all values of z.
- Therefore, the region of convergence should be mentioned along with the z-transformation.
- The Region of Convergence (ROC) of the z-transform is the set of z such that X(z) converges, i.e.,

$$\sum_{n=-\infty}^{\infty} \left| x[n] \right| r^{-n} < \infty$$

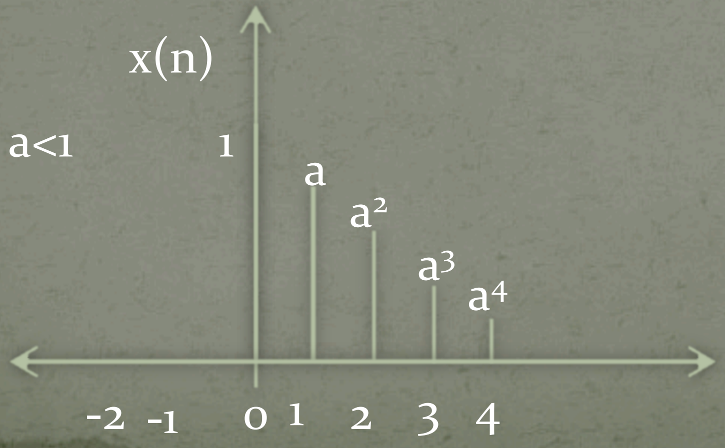
# Example #1

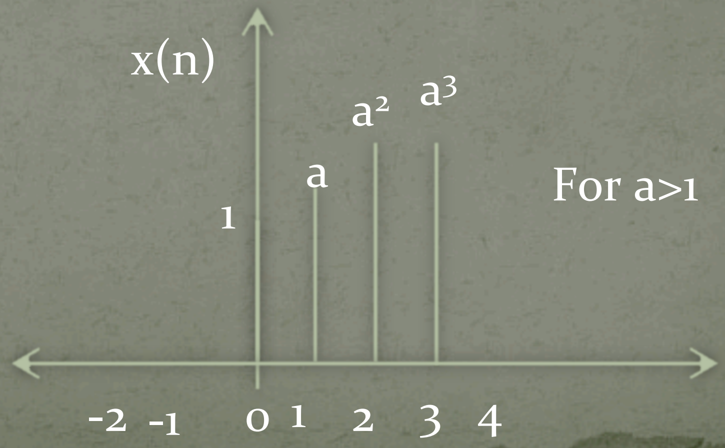
- Z-transform of right-sided exponential sequences.
- Consider the signal x[n] = $a^n$ u[n]. Mathematically it can be written as:

$$x(n) = \begin{cases} a^n & for \quad n \geq 0 \\ 0 & for \quad n < 0 \end{cases}$$

- This sequence exists for positive values of n:

For a<1

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# Example #1 (cont.)

- The z-transform of x[n] is given by:

$$X(z) = \sum_{-\infty}^{\infty} a^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

- Therefore, X(z) converges if $\sum_{n=0}^{\infty} \left(az^{-1}\right)^n < \infty$. From geometric series, we know that:

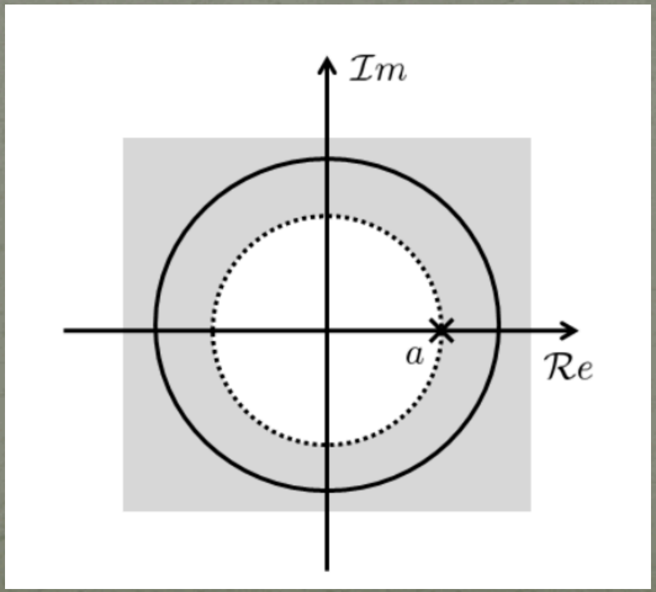
$$\sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1 - az^{-1}}$$

# Example #1 (cont.)

- When $|az^{-1}|<1$, or equivalently $|z| > |a|$. So,

$$X(z) = \frac{1}{1 - az^{-1}}$$

- With ROC being the set of z such that $|z| > |a|$. As shown below:



# Example #2

- Consider the signal: $x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$
- The z-transform is:

$$X(z) = \sum_{n=-\infty}^{\infty}\left[7\left(\frac{1}{3}\right)^n - 6\left(\frac{1}{2}\right)^n\right]u[n]z^{-n}$$

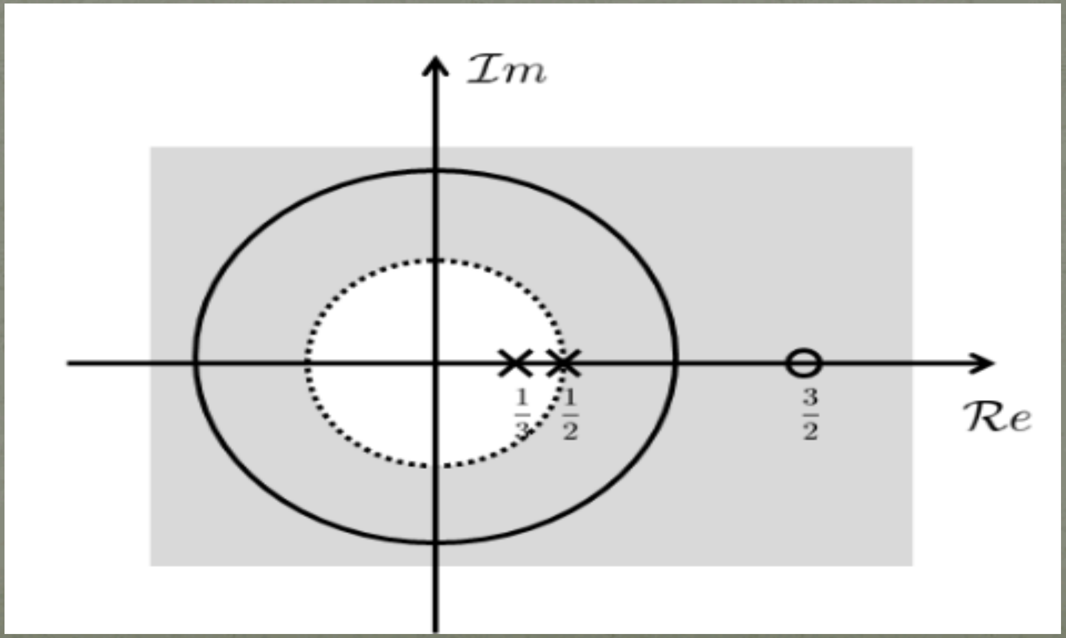
$$= 7\sum_{n=-\infty}^{\infty}\left(\frac{1}{3}\right)^n u[n]z^{-n} - 6\sum_{n=-\infty}^{\infty}\left(\frac{1}{2}\right)^n u[n]z^{-n}$$

$$= 7\left(\frac{1}{1-\frac{1}{3}z^{-1}}\right) - 6\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right)$$

$$= \frac{1-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$$

# Example #2 (cont.)

- For X(z) to converge, both sums in X(z) must converge.
- So we need both |z| > |1/3| and |z| > |1/2|. Thus the ROC is the set of z such that |z| > |1/2|.



# Properties of Z-Transform

# Linearity

- This property states that if $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$ then:

$$a_1 x_1(n) + a_2 x_2(n) \overset{z}{\longleftrightarrow} a_1 X_1(z) + a_2 X_2(z)$$

- Where $a_1$ and $a_2$ are constants.

# Time Scaling

- This property of z-transform states that if $x(n) \overset{z}{\longleftrightarrow} X(z)$ , then we can write:

$$x(n-k) \overset{z}{\longleftrightarrow} z^{-k} X(z)$$

- Where k is an integer which is shift in time in x(n) in samples.

# Scaling in Z-Domain

- This property states that is:

$$x(n) \overset{z}{\longleftrightarrow} X(z) \quad ROC: r_1 < |z| < r_2$$

$$then \quad a^n x(n) \overset{z}{\longleftrightarrow} X\left(\frac{z}{a}\right) \quad ROC: |a| r_1 < |z| < |a| r_2$$

- Where a is a constant.

# Time Reversal

- This property states that if:

$$x(n) \overset{z}{\longleftrightarrow} X(z) \qquad ROC: r_1 < |z| < r_2$$

- Then:

$$x(-n) \overset{z}{\longleftrightarrow} X(z^{-1}) \quad ROC: \frac{1}{r_1} < |z| < \frac{1}{r_2}$$

# Differentiation in Z-Domain

- This property states that if $x(n) \overset{z}{\longleftrightarrow} X(z)$ , then:

$$nx(n) \overset{z}{\longleftrightarrow} -z \frac{d\left\{X(z)\right\}}{dz}$$

# Convolution

- This property states that if $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$ then:

$$x_1(n) * x_2(n) \overset{z}{\longleftrightarrow} X_1(z) X_2(z)$$

# Example #3

- Find the convolution of sequences:

$$x_1 = \{1, -3, 2\} \quad and \quad x_2 = \{1, 2, 1\}$$

- Solution:
- Step 1: Determine z-transform of individual signal sequences:

$$X_1(z) = Z\left[x_1(n)\right] = \sum_{n=0}^{2} x_1(n)z^{-n} = x_1(0)z^0 + x_1(1)z^{-1} + x_1(2)z^{-2}$$

$$= 1z^0 - 3z^{-1} + 2z^{-2} = 1 - 3z^{-1} + 2z^{-2}$$

$$and \quad X_2(z) = Z\left[x_2(n)\right] = \sum_{n=0}^{2} x_2(n)z^{-n} = x_2(0)z^0 + x_2(1)z^{-1} + x_2(2)z^{-2}$$

$$= 1z^0 + 2z^{-1} + 1z^{-2} = 1 + 2z^{-1} + 1z^{-2}$$

# Example #3 (cont.)

- Step 2: Multiplication of $X_1(z)$ and $X_2(z)$:

$$X(z) = X_1(z)X_2(z) = \left(1 - 3z^{-1} + 2z^{-2}\right)\left(1 + 2z^{-1} + 1z^{-2}\right)$$

$$= 1 - z^{-1} - 3z^{-2} + z^{-3} + 2z^{-4}$$

- Step 3: Let us take inverse z-transform of X(z):

$$x(n) = IZT\left[1 - z^{-1} - 3z^{-2} + z^{-3} + 2z^{-4}\right] = \left\{1, -1, -3, 1, 2\right\}$$

# Other Properties of Z-Transform

- Correlation of Two Sequences:
  - This property states that if $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$ then:

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2(n-m) \overset{z}{\longleftrightarrow} X_1(z)X_2(z^{-1})$$

- Multiplication:
  - This property states that if $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$ then:

$$x_1(n)x_2(n) \overset{z}{\longleftrightarrow} \frac{1}{2\pi j} \oint_c X_1(v)X_2\left(\frac{z}{v}\right)z^{-1}\,dv$$

  - Where C is the closed contour which encloses the origin and lies in the ROC that is common to both X1(n) and X1(v) and X2 (1/v).

# Other Properties of Z-Transform (cont.)

- Conjugate of a Complex Sequence:
  - This property states that if x (n) is a complex sequence and if $x(n)\overset{z}{\longleftrightarrow}X(z)$ then:

$$x^*(n)\overset{z}{\longleftrightarrow}\left[X(z^*)\right]^*$$

- Real Part of a Sequence:
  - This property states that if x (n) is a complex sequence and if $x(n)\overset{z}{\longleftrightarrow}X(z)$ then:

$$\mathrm{Re}\left[x(n)\right]\overset{z}{\longleftrightarrow}\frac{1}{2}\left[X(z)+X^*(z^*)\right]$$

# Other Properties of Z-Transform (cont.)

- Imaginary Part of a Sequence:
  - This property of z-transform states that if x(n) is a complex sequence and if $x(n) \overset{z}{\longleftrightarrow} X(z)$ then:

$$\mathrm{Im}\left[x(n)\right] \overset{z}{\longleftrightarrow} \frac{1}{2j}\left[X(z) - X^{*}(z^{*})\right]$$

- Initial Value Theorem:
  - The initial value theorem for a single-sided sequence x(n) is x(o) whereas this is x(-∞) for double-sided sequence.
  - It is difficult to get this value from the knowledge of one-sided z-transform.
  - This theorem states that for a causal sequence x(n), x(o) can be obtained by the knowledge of X(z) i.e., one-sided of X(z) i.e.,

$$x(0) = \underset{z \to \infty}{LT} X(z)$$

# Other Properties of Z-Transform (cont.)

- Final Value Theorem:
  - the final value theorem states that if a sequence x(n) has finite value as n<∞, called as x(∞), then this value can be determined by the knowledge of its one-sided z-transform i.e.,

$$\underset{n\to\infty}{LT}\, x(n) = x(\infty) \underset{z\to 1}{LT}\left[(z-1)X(z)\right]$$

- Partial Sum:
  - It states that:

$$\sum_{-\infty}^{\infty} x(n) \leftrightarrow \frac{X(z)}{1-z^{-1}}$$

# Other Properties of Z-Transform (cont.)

- Parseval's Theorem:
  - It states that:

$$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(v)\, X_2^*\left(\frac{1}{v}\right) v^{-1}\, dv$$

  - Where $x_1(n)$ and $x_2(n)$ are complex valued sequences.

# Example #4

- Find the initial and final value of x(n) if its z-transform X(z) is given by:

$$X(z) = \frac{0.5z^2}{(z-1)(z^2-0.85z+0.35)}$$

- Solution:
  - The initial value x(0) is given by:

$$x(0) = \underset{z\to\infty}{LT} X(z) = \underset{z\to\infty}{LT} \frac{0.5z^2}{(z-1)(z^2-0.85z+0.35)} = \underset{z\to\infty}{LT} \frac{0.5z^2}{(z)(z^2)} = 0$$

  - The final value or steady value of x(n) is given by:

$$x(\infty) = \underset{z\to 1}{LT}\left[(z-1)X(z)\right] = \frac{0.5}{(1-0.85+0.35)} = 1.0$$

# Standard Z-Transform Pairs

# Z-Transform Pairs

**TABLE 3.1 SOME COMMON $z$-TRANSFORM PAIRS**

| Sequence | Transform | ROC |
|---|---|---|
| 1. $\delta[n]$ | $1$ | All $z$ |
| 2. $u[n]$ | $\dfrac{1}{1-z^{-1}}$ | $\lvert z\rvert > 1$ |
| 3. $-u[-n-1]$ | $\dfrac{1}{1-z^{-1}}$ | $\lvert z\rvert < 1$ |
| 4. $\delta[n-m]$ | $z^{-m}$ | All $z$ except 0 (if $m > 0$) or $\infty$ (if $m < 0$) |
| 5. $a^n u[n]$ | $\dfrac{1}{1-az^{-1}}$ | $\lvert z\rvert > \lvert a\rvert$ |
| 6. $-a^n u[-n-1]$ | $\dfrac{1}{1-az^{-1}}$ | $\lvert z\rvert < \lvert a\rvert$ |

# Z-Transform Pairs (cont.)

| | | |
|---|---|---|
| 7. $na^n u[n]$ | $\dfrac{az^{-1}}{(1-az^{-1})^2}$ | $\|z\| > \|a\|$ |
| 8. $-na^n u[-n-1]$ | $\dfrac{az^{-1}}{(1-az^{-1})^2}$ | $\|z\| < \|a\|$ |
| 9. $[\cos \omega_0 n]u[n]$ | $\dfrac{1-[\cos \omega_0]z^{-1}}{1-[2\cos \omega_0]z^{-1}+z^{-2}}$ | $\|z\| > 1$ |
| 10. $[\sin \omega_0 n]u[n]$ | $\dfrac{[\sin \omega_0]z^{-1}}{1-[2\cos \omega_0]z^{-1}+z^{-2}}$ | $\|z\| > 1$ |
| 11. $[r^n \cos \omega_0 n]u[n]$ | $\dfrac{1-[r\cos \omega_0]z^{-1}}{1-[2r\cos \omega_0]z^{-1}+r^2z^{-2}}$ | $\|z\| > r$ |
| 12. $[r^n \sin \omega_0 n]u[n]$ | $\dfrac{[r\sin \omega_0]z^{-1}}{1-[2r\cos \omega_0]z^{-1}+r^2z^{-2}}$ | $\|z\| > r$ |
| 13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$ | $\dfrac{1-a^N z^{-N}}{1-az^{-1}}$ | $\|z\| > 0$ |

# The End