Signal & Systems

Lecture # 7

Fourier Series

13^{th} December 18

Historical Perspective

History

- In 1822, the French mathematician J.B.J. Fourier had first studied the periodic function and published his famous theorem.
- It states that any periodic signal can be resolved into sinusoidal components.
- Fourier series is the resulting summation of harmonic sinusoid.
- The signal can be in time domain or in frequency domain.
- T can be represented either in the form of infinite trigonometric series or in the form of exponential series.

Introduction

Definition

- Fourier Series expansion is used for periodic signals to expand them in terms of their harmonics which are sinusoidal and orthogonal to one another.
- We have two types of Fourier Series expansion:
  - Continuous Time Fourier Series
  - Discrete Time Fourier Series
- Fourier Series is used for analysis of periodic signals only.
- For analysis of non-periodic signals Fourier Transform is used.

Response of LTI Systems

Response of LTI Systems to Complex Exponential

- For analyzing LTI systems, the signals can be represented as a linear combination of basic signals.
- Basic signals possess the following two properties:
  - The set of basic signals can be used to construct a broad and useful class of signals.
  - Should have simple structure in LTI system response.
- Both of these properties are provided by the set of complex exponential signals in continuous and discrete time.
- The response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude.

Response of LTI Systems to Complex Exponential (cont.)

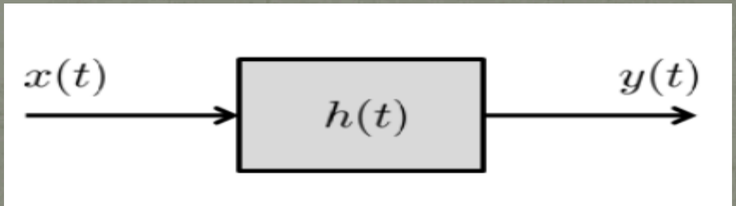
- For Continuous time: $e^{st} \rightarrow H(s)e^{st}$ where H(s) is a function of s.
- For Discrete time: $z^n \rightarrow H(z)z^n$ where H(z) is a function of z.

Eigen-functions of an LTI Systems

- If the output is a scaled version of its input, then the input function is called an Eigen-function of the system.
- The scaling factor is called the eigenvalue of the system.

Continuous Time

- Consider an LTI system with impulse response h(t) and input signal x(t).



- Suppose that x(t) = e^{st} for some s belongs to C, then the output is given by:

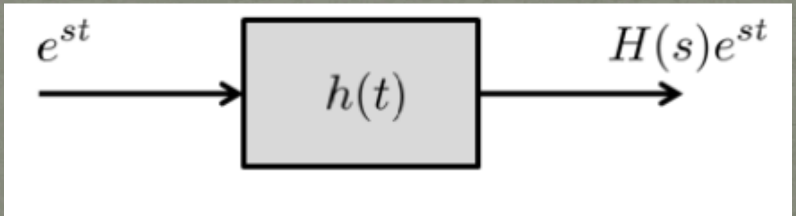
$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau$$

$$= e^{st} \left[\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \right] = H(s) e^{st} = H(s) x(t)$$

Continuous Time (cont.)

- Where H(s) is defined as: $H(s)=\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau$
- From the above derivation we see that if the input is $x(t) = e^{st}$, then the output is a scaled version $y(t) = H(s)\,e^{st}$.



- Therefore, using the definition of Eigenfunction, we show that:
  - e^{st} is an Eigenfunction of any continuous-time LTI system
  - H(s) is the corresponding eigenvalue.

Continuous Time (cont.)

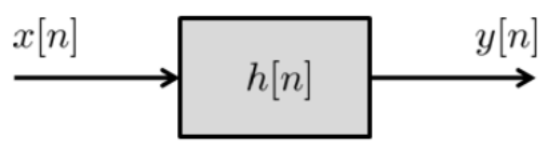
- Considering the subclass of periodic complex exponentials of the $e^{j\omega t}$, ω belongs to R by setting s=jω, then:

$$H(s)\Big|_{s=j\omega} = H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\, d\tau$$

- H(jω) is called the frequency response of the system.

Discrete Time

- In parallel manner we can show that complex exponential sequences are Eigenfunctions of discrete-time LTI systems.



- Suppose tat the impulse response is given by h[n] and the input is $x[n]=z^n$, then the output y[n] is:

$$y[n]=h[n]*x[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]$$

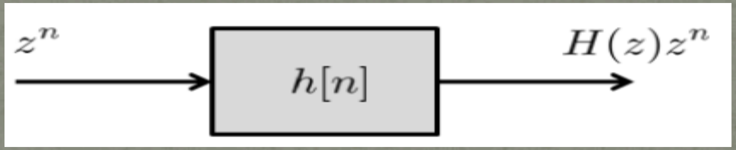
$$=\sum_{k=-\infty}^{\infty}h[k]z^{[n-k]}=z^n\sum_{k=-\infty}^{\infty}h[k]z^{-k}=H(z)z^n$$

- Where:

$$H(z)=\sum_{k=-\infty}^{\infty}h[k]z^{-k}$$

Discrete Time (cont.)

- This result indicates:
  - z^n is an Eigenfunction of a discrete-time LTI system
  - $H(z)$ is the corresponding eigenvalue.



- Considering the subclass of periodic complex exponentials $e^{-j(2\pi/N)n}$ by setting $z = e^{j2\pi/N}$, we have:

$$H(z)\big|_{z=e^{j\Omega}} = H\left(e^{j\Omega}\right) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k}$$

$$\textit{where} \quad \Omega = \frac{2\pi}{N}$$

- And $H(e^{j\Omega})$ is called the frequency response of the system.

Importance of EigenFunction

- The usefulness of Eigenfunctions can be seen from an example.
- Lets consider a signal x(t):

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$$

- According to the Eigenfunction analysis , the output of each complex exponential is:

$$e^{s_1 t} \rightarrow H(s_1) e^{s_1 t}$$

$$e^{s_2 t} \rightarrow H(s_2) e^{s_2 t}$$

$$e^{s_3 t} \rightarrow H(s_3) e^{s_3 t}$$

Importance of EigenFunction (cont.)

- From the superposition property the response to the sum is the sum of the responses, so that:

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$$

- The result implies that if the input is a linear combination of complex exponentials, the output of an LTI system is an infinite sum of complex exponentials.
- More generally, if x(t) is an infinite sum of complex exponentials,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{s_k t}$$

Importance of EigenFunction (cont.)

- Then the output is: $y(t) = \sum_{k=-\infty}^{\infty} a_k H(s_k) e^{s_k t}$
- Similarly for discrete-time signals, if:

$$x[n] = \sum_{k=-\infty}^{\infty} a_k z_k^n$$

then

$$y[n] = \sum_{k=-\infty}^{\infty} a_k H(z_k) z_k^n$$

- This is an important observation, because as long as we can express a signal x(t) as a linear combination of Eigenfunctions, then the output y(t) can be easily determined by looking at the transfer function. Same goes for discrete-time.
- The transfer function is fixed for an LTI system.

Fourier Series of Continuous-Time Periodic Signals

Fourier Series of Continuous-Time

- According to the definition of periodic signals: $x(t) = x(t+T)$ with fundamental period T and fundamental frequency $\omega_o = 2\pi/T$.
- We have also discussed two basic signals, the sinusoidal signal: $x(t)=\cos\omega_o t$ and the periodic complex exponential $x(t) = e^{j\omega o t}$.
- Both of these signals are periodic with fundamental frequency ω_o and the fundamental period $T=2\pi/\omega_o$.
- Harmonically related complex exponentials:

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t}, k = 0, \pm 1, \pm 2, \ldots\ldots$$

- Each harmonic has fundamental frequency which is multiple of ω_o .

Fourier Series of Continuous-Time (cont.)

- A Linear combination of harmonically related complex exponentials:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

- Above equation is also periodic with period T.
- k=±1 have fundamental frequency ω_o (first harmonic)
- k=±N have fundamental frequency $N\omega_o$ (Nth harmonic)

Continuous-Time Fourier Series Coefficients

- Theorem: The continuous-time Fourier series coefficients a_k of the signal:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \qquad \textit{Synthesis} \quad \textit{Equation}$$

- Is given by:

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\, dt, \qquad \textit{Analysis} \quad \textit{Equation}$$

- Proof:
- Let us consider the signal:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Continuous-Time Fourier Series Coefficients (cont.)

- If we multiply $e^{-jn\omega_0 t}$ on both sides, then we have:

$$x(t)e^{-jn\omega_0 t} = \left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right] e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}$$

- Integrating both sides from 0 to T yields: (T is the fundamental period of x(t))

$$\int_0^T x(t)e^{-jn\omega_0 t}\,dt = \int_0^T \left[\sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}\right] dt$$

$$= \sum_{k=-\infty}^{\infty} \left[a_k \int_0^T e^{j(k-n)\omega_0 t}\,dt\right]$$

Continuous-Time Fourier Series Coefficients (cont.)

- Use Euler's formula:

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt = \int_0^T \cos\left(\left(k-n\right)\omega_0 t\right)dt + j\int_0^T \sin\left(\left(k-n\right)\omega_0 t\right)dt$$

- For $k \neq n$, $\cos(k-n)\omega_o t$ and $\sin(k-n)\omega_o t$ are periodic sinusoids with fundamental period $(T/|k-n|)$
- Therefore:

$$\frac{1}{T}\int_0^T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} 1 & if \quad k = n \\ 0 & otherwise \end{cases}$$

- This result is known as the orthogonality of complex exponentials.

Continuous-Time Fourier Series Coefficients (cont.)

- Using above equation we have:

$$\int_0^T x(t)e^{-jn\omega_0 t}\,dt = Ta_n$$

- Which is equivalent to:

$$a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt$$

- Dc or constant component of x(t):

$$a_0 = \frac{1}{T}\int_T x(t)\,dt$$

Example #1

- Consider the signal: $x(t) = 1 + \frac{1}{2}\cos 2\pi t + \sin 3\pi t$
- The period of x(t) is T=2, so the fundamental frequency is $\omega_0 = 2\pi/T = \pi$.
- Recall Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$, we have:

$$x(t) = 1 + \frac{1}{4}\left[e^{j2\pi t} + e^{-j2\pi t}\right] + \frac{1}{2j}\left[e^{j3\pi t} - e^{-j3\pi t}\right]$$

$$a_0 = 1, \quad a_1 = a_{-1} = 0, \quad a_2 = a_{-2} = \frac{1}{4}, \quad a_3 = \frac{1}{2j}, \quad a_{-3} = -\frac{1}{2j}$$

$$and \quad a_k = 0 \quad otherwise$$

Conditions for Existence of Fourier Series

Existence of Fourier Series

- To understand the validity of Fourier Series representation, lets examine the problem of approximation a given periodic signal x(t) by a linear combination of a finite number of harmonically related complex exponentials.
- That is by finite series of the form:

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

- Let $e_N(t)$ denote the approximation error; i.e.,

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

Existence of Fourier Series (cont.)

- The criterion that we will use is the energy in the error over one period:

$$E_N(t) = \int_T |e_N(t)|^2 \, dt$$

- To achieve min E_N, one should define:

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} \, dt$$

- As N increases, E_N decreases and as N➔∞ E_N is zero.
- If a_k ➔∞ the approximation will diverge.
- Even for bounded a_k the approximation may not be applicable for all periodic signals.

Convergence Conditions of Fourier Series Approximation

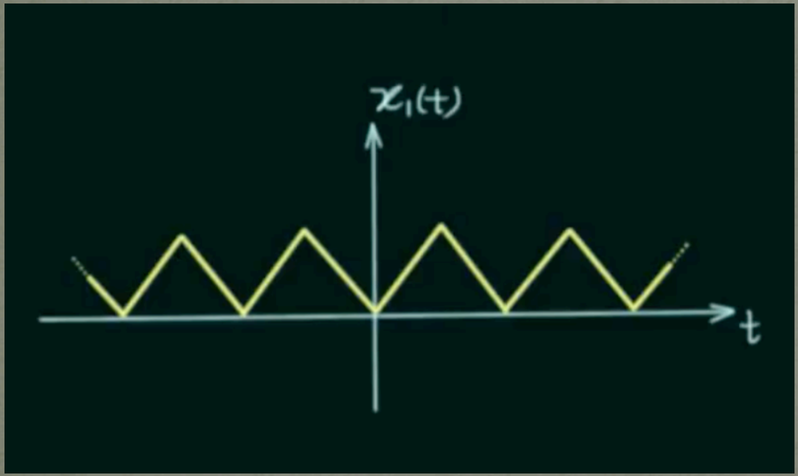
- Energy of signal should be a finite in a period:

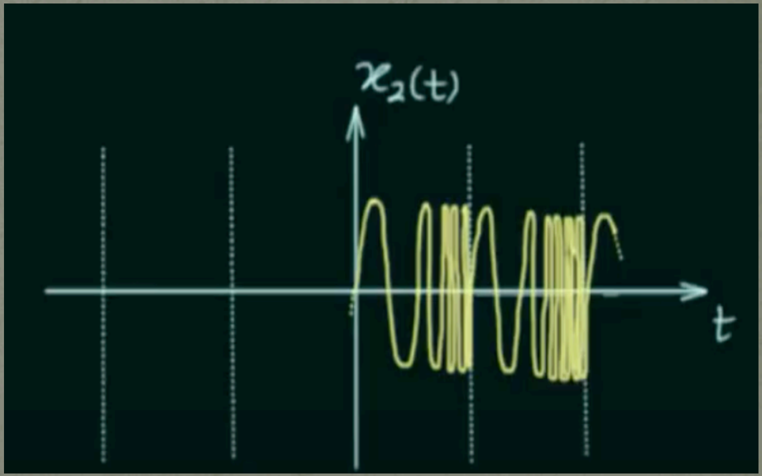
$$\int_T |x(t)|^2 \, dt < \infty$$

  - This condition only guarantees $E_N \rightarrow 0$.
  - It does not guarantee that x(t) equals to its Fourier series at each moment t.

Dirichlet Conditions

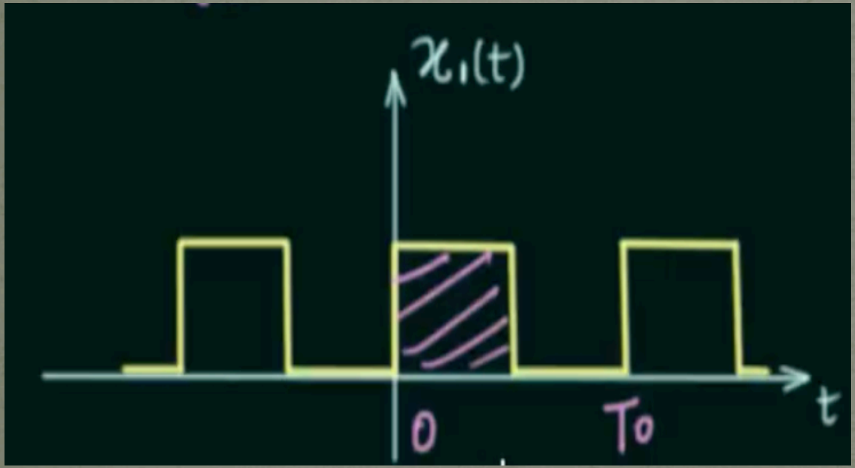
- Condition#1:
  - Signal should have finite number of maxima and minima over the range of time period.

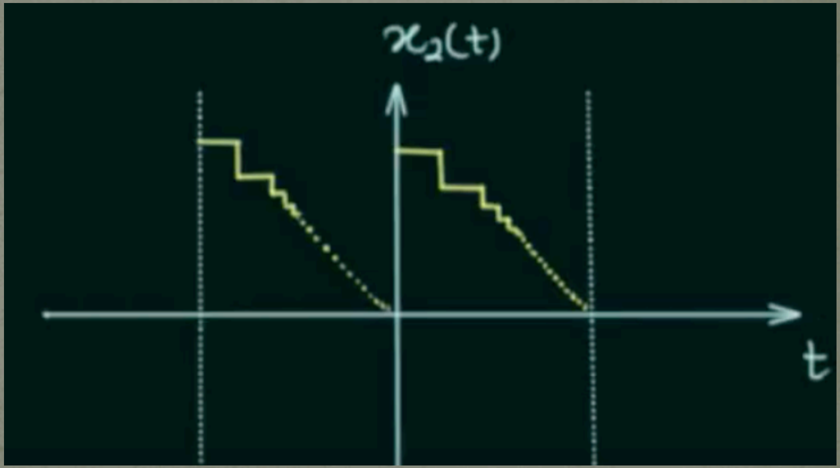




Dirichlet Conditions (cont.)

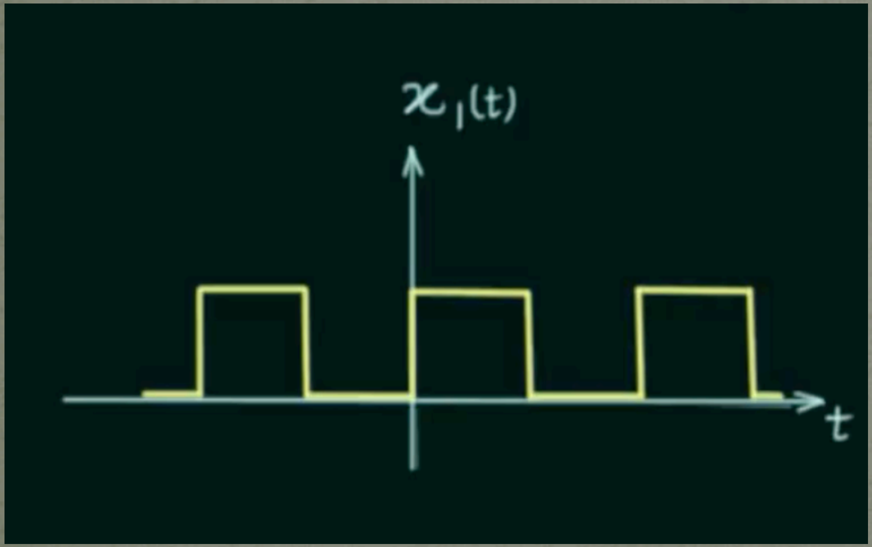
- Condition #2:
  - Signal should have finite number of discontinuities over the range of time period.

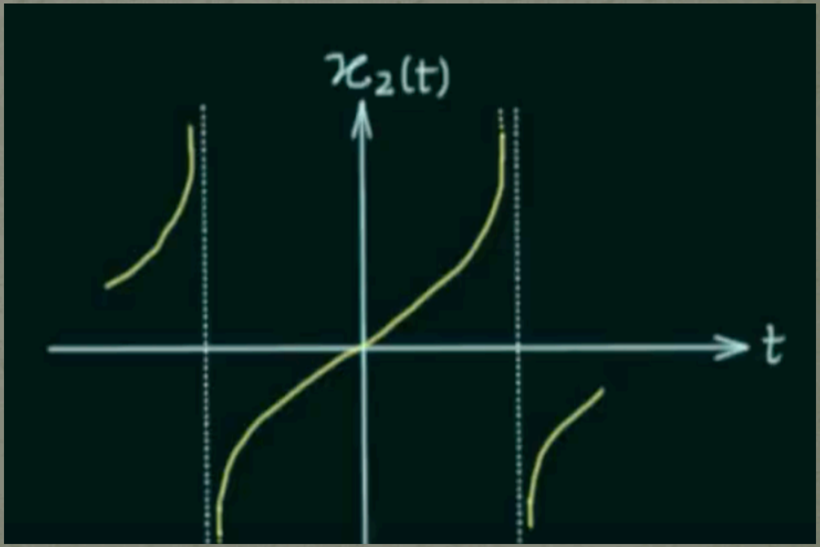




Dirichlet Conditions (cont.)

- Condition #3:
  - Signal should be absolutely integrable over the range if time period.





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