Signal & Systems

Lecture # 9

Fourier Series - II

7^{th} December 18

Properties of Fourier Series

Linearity

- For continuous-time Fourier series, we have:

$$x_1(t) \leftrightarrow a_k \quad and \quad x_2(t) \leftrightarrow b_k$$

$$Ax_1(t) + Bx_2(t) \leftrightarrow Aa_k + Bb_k$$

- For Discrete-time case, we have:

$$x_1[n] \leftrightarrow a_k \quad and \quad x_2[n] \leftrightarrow b_k$$

$$Ax_1[n] + Bx_2[n] \leftrightarrow Aa_k + Bb_k$$

Time Shift

$$x(t-t_0) \leftrightarrow a_k e^{-jk\omega_0 t_0}$$

$$x[n-n_0] \leftrightarrow a_k e^{-jk\Omega_0 n_0}$$

- Proof: Let us consider the Fourier series coefficient bk of the signal y(t)=x(t-to).

$$b_k = \frac{1}{T}\int_T x(t-t_0)e^{-j\omega_0 t}\,dt$$

- Letting τ = t-to in the integral, we obtain:

$$\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau+t_0)}\,dt = e^{-jk\omega_0 t_0}\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0 \tau}\,dt$$

$$\textit{where} \quad x(t) \leftrightarrow a_k. \quad \textit{Therefore,}$$

$$x(t-t_0) \leftrightarrow a_k e^{-jk\omega_0 t_0}$$

Time Reversal

$$x(-t) \leftrightarrow a_{-k}$$

$$x[-n] \leftrightarrow a_{-k}$$

- Proof: Consider a signal y(t) = x(-t). The Fourier series representation of x(-t) is:

$$x(-t) = \sum_{-\infty}^{\infty} a_k e^{-jk2\pi t/T}$$

- Letting k = -m, we have:

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm2\pi t/T}$$

- Thus:

$$x(-t) \leftrightarrow a_{-k}$$

Time Scaling

- Time scaling is an operation that in general changes the period of the underlying signal.
- Specifically if $x(t)$ is periodic with period T and fundamental frequency $\omega_0=2\pi/T$, then $x(\alpha t)$, where α is a positive real number, is periodic with period T/α and fundamental frequency $\alpha\omega_0$.

Multiplication

- Continous-Time Fourier Series:
  - Suppose that x(t) and y(t) are both periodic with period T and that:

$$x(t) \leftrightarrow a_k$$

$$y(t) \leftrightarrow b_k$$

  - Since the product x(t) y(t) is also periodic with period T, we can expand it in a Fourier series with Fourier series coefficients h_k expressed in terms of those for x(t) and y(t). The result is:

$$x(t)y(t) \leftrightarrow h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

  - The sum on the R.H.S may be interpreted as the Discrete-time convolution of the sequence x(t) and y(t).

Multiplication (cont.)

- Discrete-Time Fourier Series:
  - In discrete-time, suppose that:

$$x[n] \leftrightarrow a_k$$
$$and$$
$$y[n] \leftrightarrow b_k$$

  - Are both periodic with period N. then the product x[n] y[n] is also periodic with period N.
  - Its Fourier coefficients d_k are given by:

$$x[n]y[n] \leftrightarrow d_k = \sum_{l=\langle N \rangle} a_l b_{k-l}$$

  - The result is a periodic convolution between the FS sequences.
  - w[n] is periodic with N.

Conjugation & Conjugate Symmetry

- Continuous-Time Fourier Series:
  - Real $x(t) \leftrightarrow a_{-k} = a_k^*$ (conjugate symmetric)
  - Real & Even $x(t) \leftrightarrow a_k = a_k^*$ (real and even a_k)
  - Real & Odd $x(t) \leftrightarrow a_k = -a_k^*$ (purely imaginary and odd a_k), $a_0=0$
  - Even part of $x(t) \leftrightarrow \mathrm{Re}\{a_k\}$
  - Odd part of $x(t) \leftrightarrow j\,\mathrm{Im}\{a_k\}$

Parseval's Relation

- Continuous-Time Fourier Series:
  - Parseval's relation for continuous-time periodic signal is:

$$\frac{1}{T}\int_T |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

  - Where a_k are the Fourier series coefficients of x(t) and T is the period of the signal.
  - L.H.S of the above equation is the average power (i.e., energy per unit time) in one period of the periodic signal x(t).
  - Also:

$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2\, dt = |a_k|^2$$

  - So that $|a_k|^2$ is the average power in the k^{th} harmonic component of x(t).
  - Thus Parseval's relation states that the total average power in a periodic signals equals the sum of the average powers in all of its harmonic components.

Parseval's Relation (cont.)

- Discrete-Time Fourier Series:
  - Parseval's relation for discrete-time periodic signals is given by:

$$\frac{1}{N}\sum_{n=\langle N\rangle}\left|x[n]\right|^2=\sum_{n=\langle N\rangle}\left|a_k\right|^2$$

  - The average power in a periodic signal = the sum of the average power in all of its harmonic components.

First Difference

- Discrete-Time Fourier Series:
  - If x[n] is periodic with period N, then so is y[n], since shifting x[n] or linearly combining x[n] with another periodic signal whose period is N always results in a periodic signal with period N.
  - Also, if:

$$x[n] \leftrightarrow a_k$$

  - Then the Fourier coefficients corresponding to the first difference of x[n] may be expressed as:

$$x[n] - x[n-1] \leftrightarrow \left(1 - e^{-jk(2\pi/N)}\right) a_k$$

Properties of Continuous-Time Fourier Series

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamenta l frequency $\omega_0 = 2\pi / T$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	$x(t - t_0)$	$e^{-jk\omega_0 t} a_k$
Frequency shifting	$e^{jM\omega_0 t} x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(\alpha t)$, $\alpha > 0$ (Periodic with period T/α)	a_k
Periodic Convolution	$\int_T x(\tau) y(t - \tau) d\tau$	$Ta_k b_k$
Multiplication	$x(t) y(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk \frac{2\pi}{T} a_k$
Integration	$\int_{-\infty}^{t} x(t) dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right) a_k = \left(\frac{1}{jk(2\pi / T)}\right) a_k$

Properties of Continuous-Time Fourier Series (cont.)

Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathrm{Re}\{a_k\} = \mathrm{Re}\{a_{-k}\} \\ \mathrm{Im}\{a_k\} = -\mathrm{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \angle a_k = -\angle a_{-k} \end{cases}$
Real and Even Signals Real and Odd Signals Even-Odd Decomposition of Real Signals	$x(t)$ real and even $x(t)$ real and odd $\begin{cases} x_e(t) = Ev\{x(t)\} \quad [x(t)\ real] \\ x_e(t) = Od\{x(t)\} \quad [x(t)\ real] \end{cases}$	a_k real and even a_k purely imaginary and odd $\mathrm{Re}\{a_k\}$ $j\,\mathrm{Im}\{a_k\}$
	Parseval's Relation for Periodic Signals $\frac{1}{T}\int_T \lvert x(t) \rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert a_k \rvert^2$	

Properties of Discrete-Time Fourier Series

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamenta l frequency $\omega_0 = 2\pi$	$\begin{array}{l} a_k \\ b_k \end{array}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$e^{-jk(2\pi/N)t} a_k$
Frequency shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}

Properties of Discrete-Time Fourier Series (cont.)

Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & if\ n\ is\ a\ multiple\ of\ n \\ 0, & if\ n\ is\ a\ multiple\ of\ n \end{cases}$ (Periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix} viewed\ as\ periodic \\ with\ period\ mN \end{pmatrix}$
Periodic Convolution	$\sum_{r=[N]} x[r]y[n-r]$	$N a_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=<N>} a_l b_{k-l}$
Differentiation	$x[n]-x[n-1]$	$\left(1-e^{-jk(2\pi/N)}\right)a_k$
Integration	$\sum_{k=-\infty}^{n} x[k]$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{1-e^{-jk(2\pi/N)}}\right)a_k$

Properties of Discrete-Time Fourier Series (cont.)

Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathrm{Re}\{a_k\} = \mathrm{Re}\{a_{-k}\} \\ \mathrm{Im}\{a_k\} = -\mathrm{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \angle a_k = -\angle a_{-k} \end{cases}$
Real and Even Signals Real and Odd Signals Even-Odd Decomposition of Real Signals	$x[n]$ real and even $x[n]$ real and odd $\begin{cases} x_e[n] = Ev\{x[n]\} & [x[n]\ real] \\ x_e[n] = Od\{x[n]\} & [x[n]\ real] \end{cases}$	a_k real and even a_k purely imaginary and odd $\mathrm{Re}\{a_k\}$ $j\,\mathrm{Im}\{a_k\}$
	Parseval's Relation for Periodic Signals $\frac{1}{T}\sum_{n=<N>} \lvert x[n] \rvert^2 = \sum_{n=<N>} \lvert a_k \rvert^2$	

Example #1

- Find C'_n in terms of C_n, where $x(t) = C_n$ and $y(t) = C'_n$.
  - (1): $y(t) = x(t+1) + x(t-1)$
  - (2): $y(t) = e^{-j2\omega_0 t}.x(t)$

Fourier Series & LTI Systems

LTI System

- The response of a continuous-time LTI system with impulse response h(t) to a complex exponential signal e^{st} is the same complex exponential multiplied by a complex gain:

$$y(t) = H(s)e^{st}$$

$$\text{where}$$

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\, d\tau$$

- In particular, for s=jω, the output is $y(t)=H(j\omega)e^{j\omega t}$.
- The complex functions H(s) and H(jω) are called the system function (or transfer function) and the frequency response, respectively.

LTI System (cont.)

- By superposition, the output of an LTI system to a periodic signal represented by a Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \quad is \quad given \quad by$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

- That is, the Fourier series coefficients b_k of the periodic output y(t) are given by:

$$b_k = a_k H(jk\omega_0)$$

- Similarly, for discrete time signals and systems, response h[n] to a complex exponential signal $e^{j\omega n}$ is the same complex exponential multiplied by a complex gain:

$$y[n] = H(jk\omega_0) e^{jk\omega_0 n}$$

$$where$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

Example #2

- Consider a causal LTI system whose i/p and o/p are related by following differential equation:

$$\frac{dy(t)}{dt} + 4y(t) = x(t)$$

- Find the o/p coefficient C'_n if i/p is $\cos 2\pi t + \sin 4\pi t$

Example #3

- x[n]=real and odd periodic signal. Where N=7 and a_k=?.

$$a_{15} = j \quad , \quad a_{16} = 2j \quad , \quad a_{17} = 3j$$

$$a_0 = ? \quad , \quad a_{-1} = a_{-2} = a_{-3} = ?$$

The End