Signal & Systems

Lecture # 10

Discrete Time Fourier Transform-II

3rd January 19

Properties of DT Fourier Transform

Differencing & Accumulation

- If:

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

- Then:

$$x[n]-x[n-1] \overset{F}{\longleftrightarrow} \left(1-e^{-j\omega}\right)X\left(e^{j\omega}\right)$$

$$For \quad signal,$$

$$y[n]=\sum_{m=-\infty}^{n} x[m],$$

- Its Fourier transform is given as:

$$\sum_{m=-\infty}^{n} x[m] \overset{F}{\longleftrightarrow} \frac{1}{\left(1-e^{-j\omega}\right)}X(e^{j\omega})+\pi X(e^{j0})\sum_{m=-\infty}^{+\infty}\delta(\omega-2\pi k)$$

Time Reversal

- If:

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

- Then:

$$x[-n] \overset{F}{\longleftrightarrow} X\left(-e^{j\omega}\right)$$

Differentiation in Frequency

- If:

$$x\left[n\right] \leftrightarrow X\left(e^{j\omega}\right)$$

- Then:

$$nx\left[n\right] \overset{F}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

Parseval's Relation

- If:

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

- Then:

$$\sum_{n=-\infty}^{+\infty} \left|x[n]\right|^2 = \frac{1}{2\pi}\int_{2\pi} \left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

Convolution Property

- If x[n], h[n] and y[n] are the input, impulse response, and output respectively, of an LTI system, so that,

$$y[n] = x[n] * h[n]$$

$$then,$$

$$Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) H\left(e^{j\omega}\right)$$

- Where $X(e^{j\omega})$, $H(e^{j\omega})$ and $Y(e^{j\omega})$ are the Fourier transforms of x[n], h[n] and y[n] respectively.

Multiplication Property

- It states that:

$$y[n] = x_1[n]x_2[n] \overset{F}{\longleftrightarrow} Y\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{2\pi} X_1\left(e^{j\omega}\right)X_2\left(e^{j(\omega-\theta)}\right)d\theta$$

Example #1

- Consider the signal: $x[n] = \delta[n] + \delta[n-1] + \delta[n+1]$
- Solution:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\delta[n] + \delta[n-1] + \delta[n+1]\right) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} + \sum_{n=-\infty}^{\infty} \delta[n-1] e^{-j\omega n} + \sum_{n=-\infty}^{\infty} \delta[n+1] e^{-j\omega n}$$

$$X\left(e^{j\omega}\right) = 1 + e^{-j\omega} + e^{j\omega} = 1 + 2\cos\omega$$

Example #2

- Given that x[n] has Fourier transform $X(e^{j\omega})$, express the Fourier transforms of the following signals in terms of $X(e^{j\omega})$, by using the Fourier transform properties:
  - (a): $x[n] = x[1-n] + x[-1-n]$
  - (b): $x[n] = (n-1)^2 x[n]$

Properties of DTFT

PROPERTY	SEQUENCE	FOURIER TRANSFORM
	$x[n]$	$X(\Omega)$
	$x_1[n]$	$X_1(\Omega)$
	$x_2[n]$	$X_2(\Omega)$
Periodicity	$x[n]$	$X(\Omega + 2\pi) = X(\Omega)$
Linearity	$a_1 x_1[n] + a_2 x_2[n]$	$a_1 X_1(\Omega) + a_2 X_2(\Omega)$
Time shifting	$x[n - n_0]$	$e^{-j\Omega n0} X(\Omega)$
Frequency shifting	$e^{j\Omega_0 n} x[n]$	$X(\Omega - \Omega_0)$
Conjugation	$x^*[n]$	$X^*(-\Omega)$
Time reversal	$x[-n]$	$X(-\Omega)$
Time scaling	$x_{(m)}[n] = \begin{cases} x[n/m] & \text{if } n = km \\ 0 & \text{if } n \neq km \end{cases}$	$X(m\Omega)$
Frequency differentiation	$nx[n]$	$j\dfrac{dX(\Omega)}{d\Omega}$
First difference	$x[n] - x[n-1]$	$(1 - e^{-j\Omega})X(\Omega)$

Properties of DTFT (cont.)

Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\pi X(0)\delta(\Omega) + \frac{1}{1-e^{-j\Omega}} X(\Omega)$ $\lvert\Omega\rvert \le \pi$
Convolution	$x_1[n] * x_2[n]$	$X_1(\Omega)X_2(\Omega)$
Multiplication	$x_1[n]x_2[n]$	$\frac{1}{2\pi} X_1(\Omega) \otimes X_2(\Omega)$
Real sequence	$x[n] = x_e[n] + x_o[n]$	$X(\Omega) = A(\Omega) + jB(\Omega)$ $X(-\Omega) = X^*(\Omega)$
Even component	$x_e[n]$	$\text{Re}\{X(\Omega)\} = A(\Omega)$
Odd component	$x_o[n]$	$j\,\text{Im}\{X(\Omega)\} = jB(\Omega)$
Parseval's theorem		

$$\sum_{n=-\infty}^{\infty} x_1[n]x_2[n] = \frac{1}{2\pi}\int_{2\pi} X_1(\Omega)X_2(-\Omega)\, d\Omega$$

$$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(\Omega\rvert^2\, d\Omega$$

Common Fourier Transform Pairs

$x[n]$	$X(\Omega)$
$\delta[n]$	1
$\delta(n-n_0)$	$e^{-j\Omega n_0}$
$x[n]=1$	$2\pi\delta(\Omega), \|\Omega\|\leq\pi$
$e^{j\Omega_0 n}$	$2\pi\delta(\Omega-\Omega_0), \|\Omega\|, \|\Omega_0\|\leq\pi$
$\cos\Omega_0 n$	$\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)], \|\Omega\|, \|\Omega_0\|\leq\pi$
$\sin\Omega_0 n$	$-j\pi[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)], \|\Omega\|, \|\Omega_0\|\leq\pi$
$u[n]$	$\pi\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}, \|\Omega\|\leq\pi$
$-u[-n-1]$	$-\pi\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}, \|\Omega\|\leq\pi$
$a^n u[n], \|a\|<1$	$\dfrac{1}{1-ae^{-j\Omega}}$
$-a^n u[-n-1], \|a\|>1$	$\dfrac{1}{1-ae^{-j\Omega}}$

Common Fourier Transform Pairs (cont.)

	$1 - ae$
$(n+1)\,a^n u[n], \lvert a \rvert < 1$	$\dfrac{1}{\left(1 - ae^{-j\Omega}\right)^2}$
$a^{\lvert n \rvert}, \lvert a \rvert < 1$	$\dfrac{1 - a^2}{1 - 2a\cos\Omega + a^2}$
$x[n] = \begin{cases} 1 & \lvert n \rvert \le N_1 \\ 0 & \lvert n \rvert > N_1 \end{cases}$	$\dfrac{\sin\left[\Omega\left(N_1 + \frac{1}{2}\right)\right]}{\sin(\Omega/2)}$
$\dfrac{\sin Wn}{\pi n}, 0 < W < \pi$	$X(\Omega) = \begin{cases} 1 & 0 \le \lvert \Omega \rvert \le W \\ 0 & W < \lvert \Omega \rvert \le \pi \end{cases}$
$\displaystyle\sum_{k=-\infty}^{\infty} \delta[n - kN_0]$	$\displaystyle\Omega_0 \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_0), \Omega_0 = \frac{2\pi}{N_0}$

Systems Characterized by Linear Constant-Coefficient Difference Equations

Linear Constant-Coefficient Difference Equations

- A general linear constant-coefficient difference equation for an LTI system with input x[n] and output y[n] is of the form,

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- Which is usually referred to as Nth-order difference equation.
- If $x[n] = e^{j\omega n}$ is the input to an LTI system, then the output must be of the form $H(e^{j\omega})e^{j\omega n}$. Substituting these expressions into above equation and performing some algebra allow us to solve for $H(e^{j\omega})$.

Linear Constant-Coefficient Difference Equations (cont.)

- Based on convolution, above equation can be written as:

$$H\left(e^{j\omega}\right)=\frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)}$$

- Applying the Fourier transform to both sides and using the linearity and time-shifting properties we obtain the following expression:

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y\left(e^{j\omega}\right)=\sum_{k=0}^{M} b_k e^{-jk\omega} X\left(e^{j\omega}\right)$$

Linear Constant-Coefficient Difference Equations (cont.)

- Or equivalently

$$H\left(e^{j\omega}\right)=\frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)}=\frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

- The frequency response of the LTI system can be written down by inspection as well.

Example #3

- Consider a causal and stable LTI system S whose input x[n] and output y[n] are related through the second-order difference equation:

$$y[n]-\frac{1}{6}y[n-1]-\frac{1}{6}y[n-2]=x[n]$$

- (a): Determine the frequency response $H(e^{j\omega})$ for the system S.
- (b): Determine the impulse response h[n] for the system S.

The End