Signal & Systems

Lecture # 11

Z-Transform

10^{th} January 19

Z-Transform

The Z-Transform

- The z-transform of a sequence x[n] is: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$
- The z-transform can also be thought of as an operator Z{.} that transforms a sequence to a function:

$$Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = X(z)$$

- In both cases z is a continuous complex variable.
- The z-transform operation is denoted as: $x(n) \overset{z}{\longleftrightarrow} X(z)$
- Where "z" is the complex number. Therefore, we may write z as:

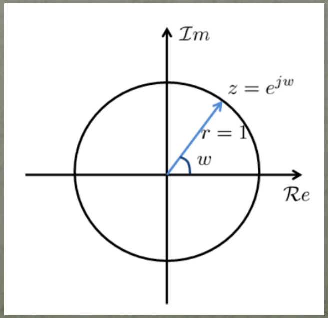
$$z = re^{j\omega}$$

The Z-Transform (cont.)

- Where r and ω belongs to Real number. When r=-1, the z-transform of a discrete-time signal becomes:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Therefore, the DTFT is a special case of the z-transform.
- Pictorially, we can view DTFT as the z-transform evaluated on the unit circle:



The Z-Transform (cont.)

- When r≠1, the z-transform is equivalent to:

$$X\left(re^{j\omega}\right)=\sum_{n=-\infty}^{\infty}x[n]\left(re^{j\omega}\right)^{-n}$$

$$=\sum_{n=-\infty}^{\infty}\left(r^{-n}x[n]\right)e^{-j\omega n}$$

$$=F\left[r^{-n}x[n]\right]$$

- Which is the DTFT of the signal r^{-n} x[n].
- However, we know that the DTFT does not always exist. It exists only when the signal is square summable, or satisfies the Dirichlet conditions.
- Therefore, X(z) does not always converge. It converges only for some values of r. this range of r is called the region of convergence (ROC).

Region of Convergence (ROC)

- The region of convergence are the values of for which the z-transform converges.
- Z-transform is an infinite power series which is not always convergent for all values of z.
- Therefore, the region of convergence should be mentioned along with the z-transformation.
- The Region of Convergence (ROC) of the z-transform is the set of z such that X(z) converges, i.e.,

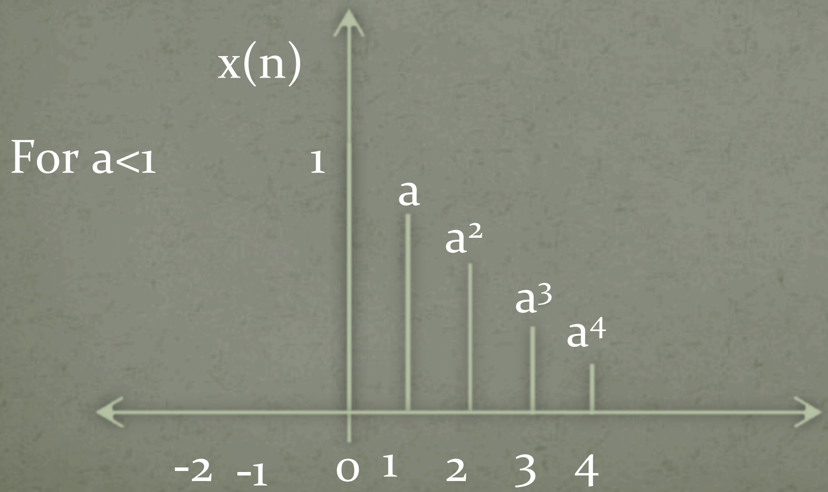
$$\sum_{n=-\infty}^{\infty} |x[n]| r^{-n} < \infty$$

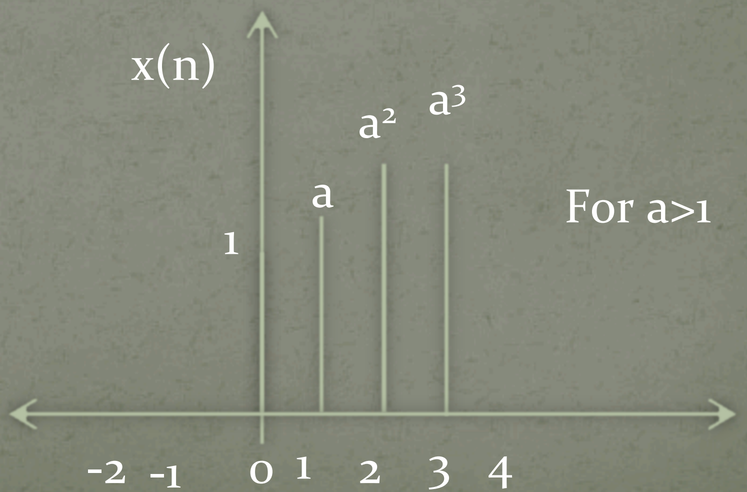
Example #1

- Z-transform of right-sided exponential sequences.
- Consider the signal $x[n] = a^n\ u[n]$. Mathematically it can be written as:

$$x(n) = \begin{cases} a^n & for \quad n \geq 0 \\ 0 & for \quad n < 0 \end{cases}$$

- This sequence exists for positive values of n:





Example #1 (cont.)

- The z-transform of x[n] is given by: $X(z)=\sum_{-\infty}^{\infty} a^n u[n] z^{-n}$

$$=\sum_{n=0}^{\infty}\left(az^{-1}\right)^n$$

- Therefore, X(z) converges if $\sum_{n=0}^{\infty}\left(az^{-1}\right)^n<\infty$. From geometric series, we know that:

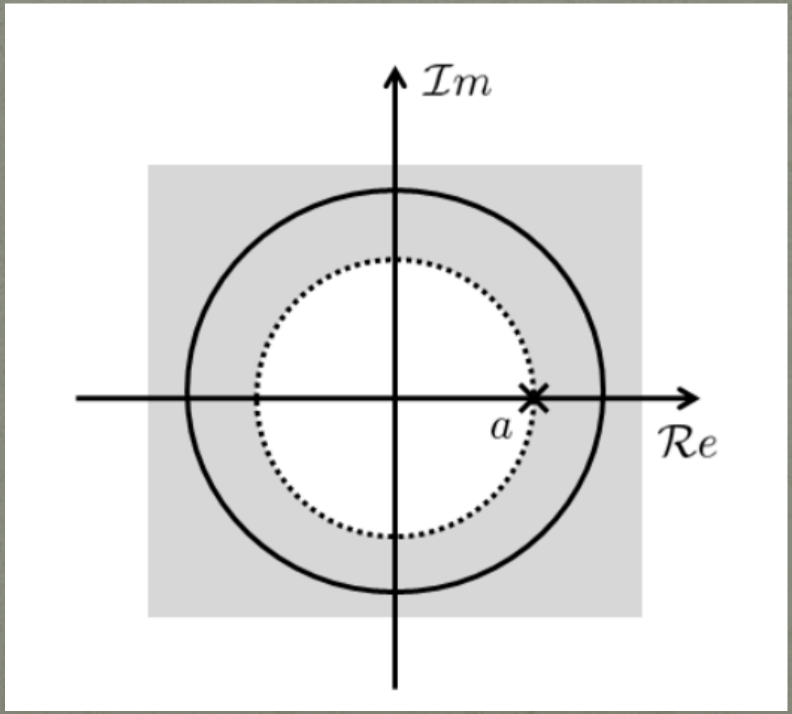
$$\sum_{n=0}^{\infty}\left(az^{-1}\right)^n=\frac{1}{1-az^{-1}}$$

- When $|az^{-1}|<1$, or equivalently $|z| > |a|$. So,

$$X(z)=\frac{1}{1-az^{-1}}$$

Example #1 (cont.)

- With ROC being the set of z such that |z| > |a|. As shown below:

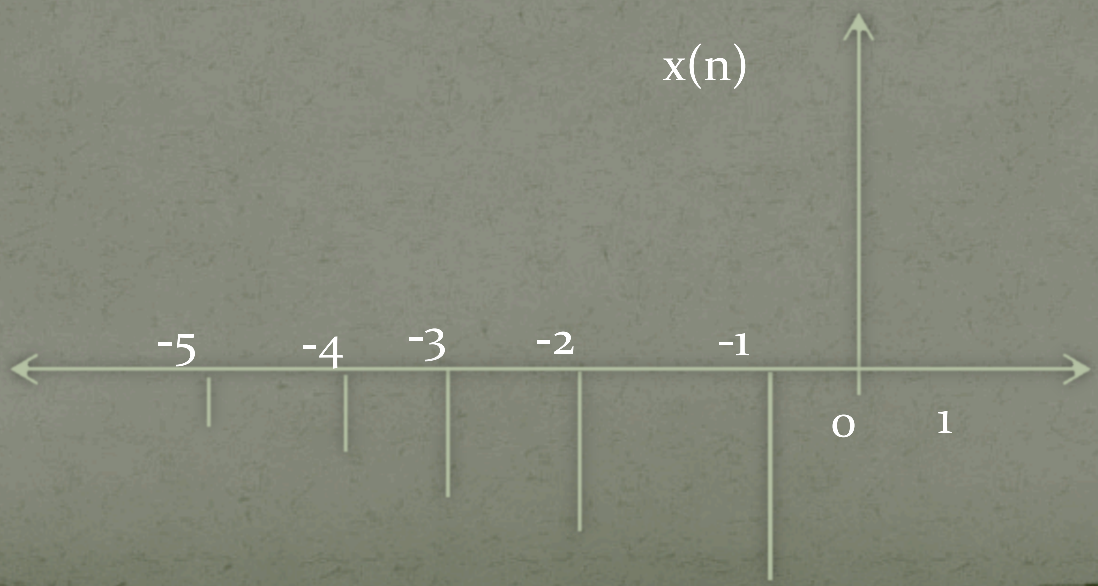


Example #2

- Consider the signal $x[n] = -a^n u[-n -1]$ with $0 < a < 1$. Which is the left sided exponential sequence and it can be determined mathematically as:

$$x(n) = \begin{cases} -a^n u(-n-1) & for \quad n \le 0 \\ 0 & for \quad n > 0 \end{cases}$$

- This sequence exists only for negative values of n.



Example #2 (cont.)

- The z-transform of x[n] is: $X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n}$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n$$

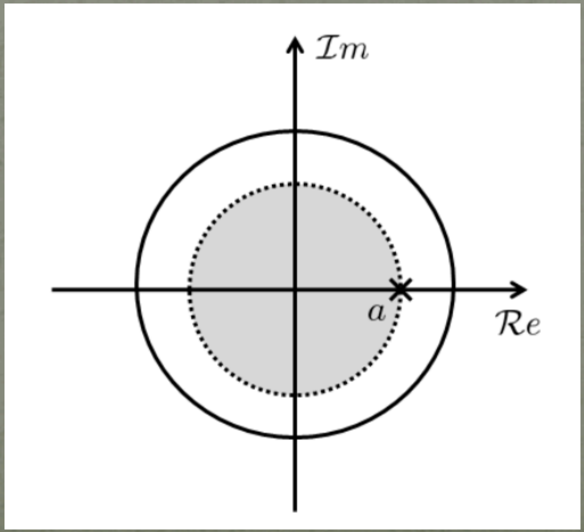
$$= 1 - \sum_{n=0}^{\infty} \left(a^{-1} z\right)^n$$

- Therefore, X(z) converges when $|a^{-1}z| < 1$, or equivalently $|z| < |a|$. In this case:

$$X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}}$$

Example #2 (cont.)

- Witch ROC being the set of z such that $|z| < |a|$.
- Note that the z-transform is the same as that of Example 1, the only difference is the ROC. Which is shown below as:



Properties of ROC

Properties of ROC

- **Property #1:** The ROC is a ring or disk in the z-plane center at origin.
- **Property #2:** The Fourier transform of x|n| converges absolutely if and only if the ROC of the z-transform includes the unit circle.
- **Property #3:** The ROC contains no poles.
- **Property #4:** If x|n| is a finite impulse response (FIR), then the ROC is the entire z-plane.
- **Property #5:** If x|n| is a right-sided sequence then the ROC extends outwards from the outermost finite pole to infinity.

Properties of ROC

- **Property #6:** If x|n| is left sided then the ROC extends inward from the innermost nonzero pole to z=0.
- **Property #7:** If X(z) is rational, i.e., X(z)=A(z) / B(z) where A(z) and B(z) are polynomials, and if x[n] is right-sided, then the ROC is the region outside the outermost pole.

Properties of Z-Transform

Linearity

- This property states that if $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$,then

$$a_1 x_1(n) + a_2 x_2(n) \overset{z}{\longleftrightarrow} a_1 X_1(z) + a_2 X_2(z)$$

- Where a_1 and a_2 are constants.

Time Scaling

- This property of z-transform states that if $x(n) \overset{z}{\longleftrightarrow} X(z)$,then we can write:

$$x(n-k) \overset{z}{\longleftrightarrow} z^{-k} X(z)$$

- Where k is an integer which is shift in time in x(n) in samples.

Scaling in Z-Domain

- This property states that if:

$$x(n) \overset{z}{\longleftrightarrow} X(z) \quad ROC: r_1 < |z| < r_2$$

$$then \quad a^n x(n) \overset{z}{\longleftrightarrow} X\left(\frac{z}{a}\right) \quad ROC: |a| r_1 < |z| < |a| r_2$$

- Where a is a constant.

Time Reversal

- This property states that if:

$$x(n) \overset{z}{\longleftrightarrow} X(z) \qquad ROC: r_1 < |z| < r_2$$

- Then,

$$x(-n) \overset{z}{\longleftrightarrow} X(z^{-1}) \quad ROC: \frac{1}{r_1} < |z| < \frac{1}{r_2}$$

Differentiation in Z-Domain

- This property states that if $x(n) \overset{z}{\longleftrightarrow} X(z)$, then:

$$nx(n) \overset{z}{\longleftrightarrow} -z\frac{d\left\{X(z)\right\}}{dz}$$

Convolution

- This property states that :

$$x_1(n) * x_2(n) \overset{z}{\longleftrightarrow} X_1(z)X_2(z)$$

Example #3

- Find the convolution of sequences: $x_1 = \{1,-3,2\} \quad and \quad x_2 = \{1,2,1\}$
- Solution:
- Step 1: Determine z-transform of individual signal sequences:

$$X_1(z) = Z[x_1(n)] = \sum_{n=0}^{2} x_1(n)z^{-n} = x_1(0)z^0 + x_1(1)z^{-1} + x_1(2)z^{-2}$$

$$= 1z^0 - 3z^{-1} + 2z^{-2} = 1 - 3z^{-1} + 2z^{-2}$$

$$and \quad X_2(z) = Z[x_2(n)] = \sum_{n=0}^{2} x_2(n)z^{-n} = x_2(0)z^0 + x_2(1)z^{-1} + x_2(2)z^{-2}$$

$$= 1z^0 + 2z^{-1} + 1z^{-2} = 1 + 2z^{-1} + 1z^{-2}$$

Example #3 (cont.)

- Step 2: Multiplication of $X_1(z)$ and $X_2(z)$:

$$X(z) = X_1(z)X_2(z) = \left(1 - 3z^{-1} + 2z^{-2}\right)\left(1 + 2z^{-1} + 1z^{-2}\right)$$

$$= 1 - z^{-1} - 3z^{-2} + z^{-3} + 2z^{-4}$$

- Step 3: Let us take inverse z-transform of X(z):

$$x(n) = IZT\left[1 - z^{-1} - 3z^{-2} + z^{-3} + 2z^{-4}\right] = \{1, -1, -3, 1, 2\}$$

Inverse Z-Transform

Inverse Z-Transform

- The inverse Z-transform is as follows:

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} \, dz$$

- Methods to obtain Inverse Z-transform:
  - If X(z) is rational, we can use expanding the rational algebraic into a linear combination of lower order terms and then one may use:
    - If ROC is out of pole $z = a_i$:

$$X(z) = \frac{A_i}{1 - a_i z^{-1}} \rightarrow x[n] = A_i a_i u[n]$$

    - If ROC is inside of $z = a_i$:

$$X(z) = \frac{A_i}{1 - a_i z^{-1}} \rightarrow x[n] = -A_i a_i u[-n-1]$$

    - Do not forget to consider ROC in obtaining inverse of ZT.

Inverse Z-Transform (cont.)

- If X(z) is non-rational, use Power series expansion of X(z), then apply $\delta[n+n_o] \leftrightarrow z^{no}$
- If X(z) is rational, power series can be obtained by long division.
  - If X(z) is a rational function of z, i.e., a ratio of polynomials, we can also use partial fraction expansion to express X(z) as a sum of simple terms for which the inverse transform may be recognized by inspection.
  - The ROC plays a critical role in this process.

Example #4

- Consider:

$$X(z) = \frac{1}{1 - az^{-1}}, |z| > |a|$$

- Expand in a power series by long division.

Properties of LTI Systems

Causality

- A discrete-time LTI system is causal if and only if ROC is the exterior of a circle (including ∞).
- Proof:
- A system is causal if and only if: $h[n] = 0, n<0$.
- Therefore, h[n] must be right sided. Property 5 implies that ROC is outside a circle.
- Also, by the definition that:

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

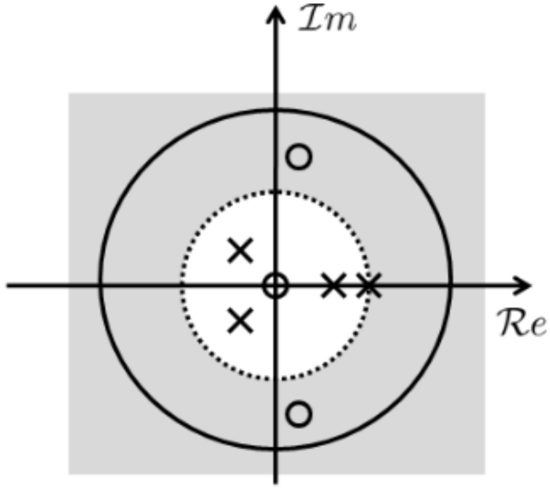
Causality (cont.)

- Where there is no positive powers of z, H(z) converges also when $z \rightarrow \infty$.
- $|z|>1$ when $z \rightarrow \infty$.

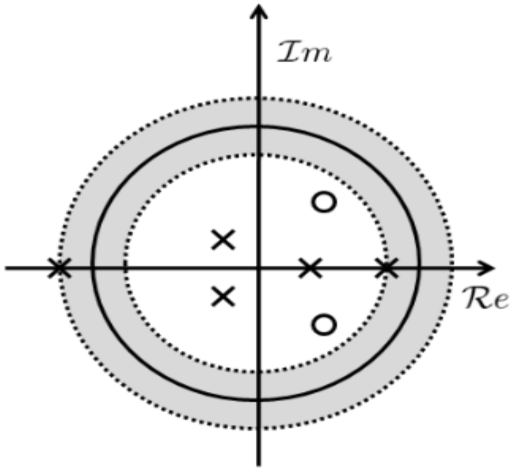
Stability

- A discrete time LTI system is stable if and only if ROC of H(z) includes the unit circle.
- Proof: A system is stable if and only if h[n] is absolutely summable, if and only if DTFT of h[n] exists. Consequently by property 2, ROC of H(z) must include the unit circle.
- A causal discrete-time LTI system is stable if and only if all of its poles are inside the unit circle.

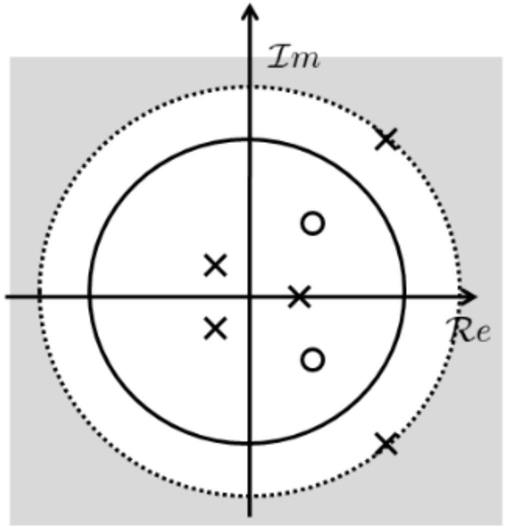
Examples



Causal, Stable



Not causal, Stable



Causal, Unstable

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