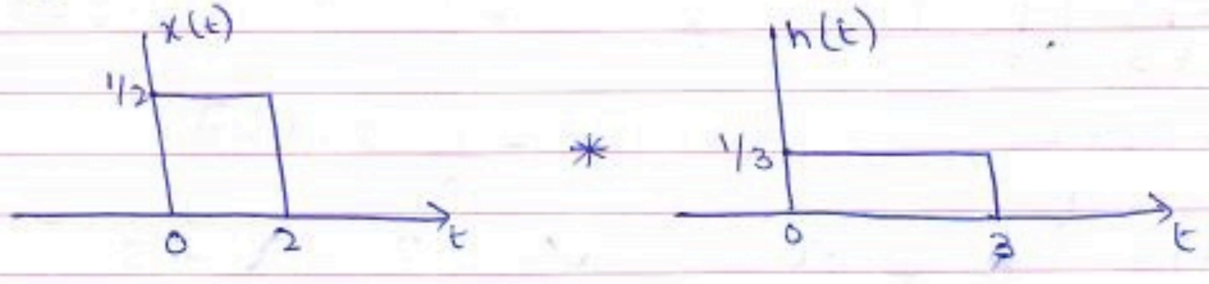
day / date: Fri / 11-01-18

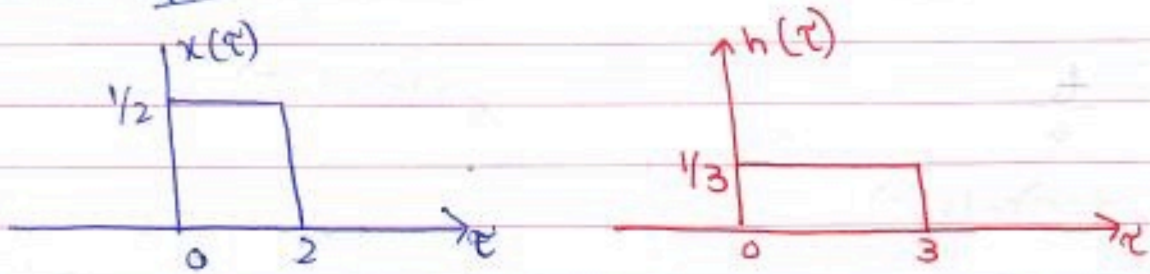
LECTURE #14
: SOLVED EXAMPLE

EXAMPLE #1:-

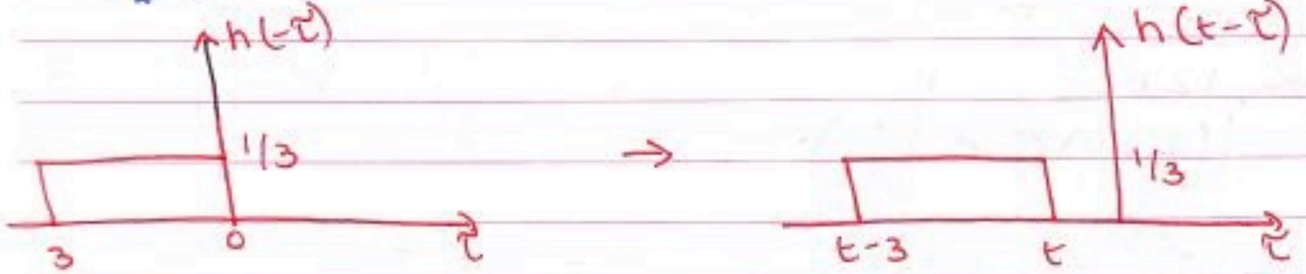


Sol:-

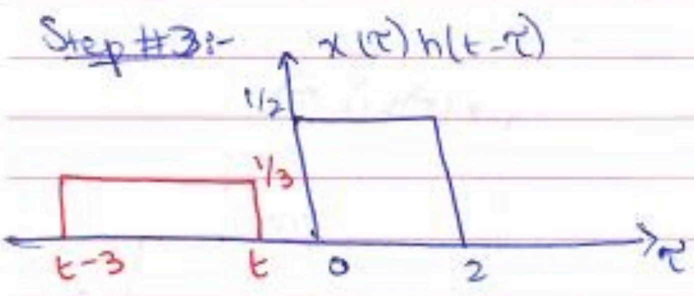
Step #1:-



Step #2:-

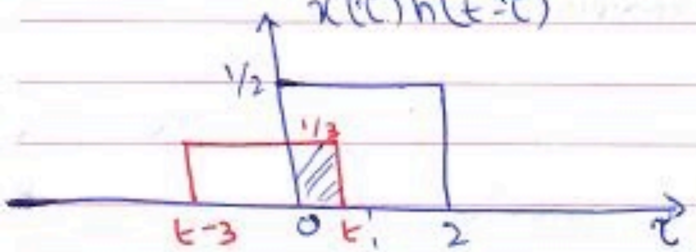


Step #3:-



$t < 0$

$y(t) = 0$ as there is no overlapping.

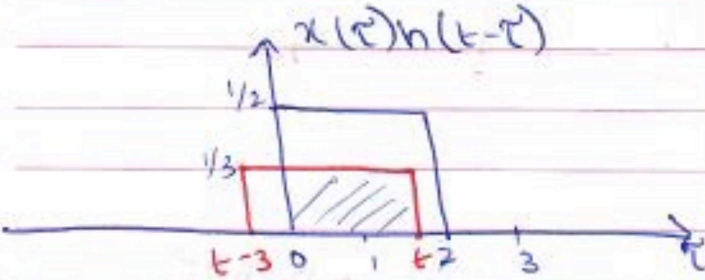


when $0 < t \le 1$

$$y(t) = \int_0^t \left(\frac{1}{2} \times \frac{1}{3}\right) d\tau$$

$$= \frac{1}{6}\int_0^t d\tau = \frac{1}{6}\left[t - 0\right]$$

$$y(t) = \frac{t}{6}$$



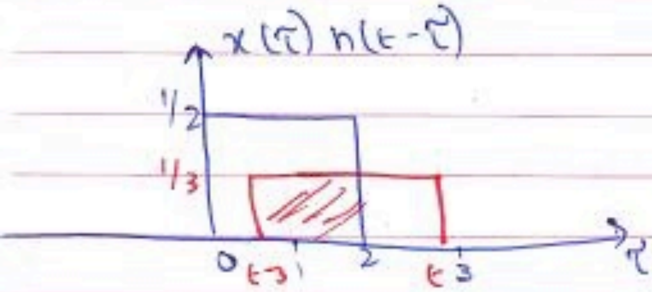
when, $1 < t \le 2$

$$y(t) = \int_0^1 \left(\frac{1}{2} \times \frac{1}{3}\right) d\tau + \int_1^t \left(\frac{1}{6}\right) d\tau$$

$$= \frac{1}{6}\left[\tau\right]_0^1 + \frac{1}{6}\left[\tau\right]_1^t$$

$$= \frac{1}{6}\left[1 - 0\right] + \frac{1}{6}\left[t - 1\right]$$

$$= \frac{\cancel{1}}{6} + \frac{1}{6}t - \frac{\cancel{1}}{6} \Rightarrow \frac{t}{6}$$



when $2 < t \le 3$

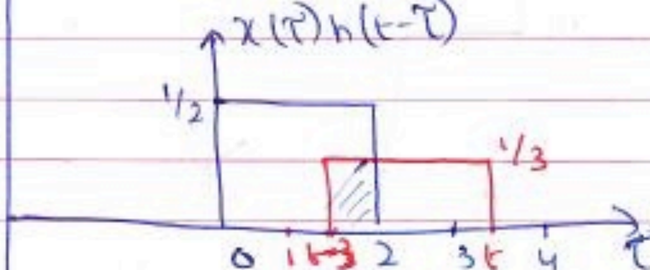
$$y(t) = \int_{t-3}^{2} \left(\frac{1}{6}\right) d\tau$$

$$= \frac{1}{6}\left[\tau\right]_{t-3}^{2} = \frac{1}{6}\left[2 - (t-3)\right]$$

$$= \frac{1}{6}\left[2 - t + 3\right]$$

$$y(t) = \frac{\cancel{2}}{\cancel{6}\,3} - \frac{t}{6} + \frac{\cancel{3}}{\cancel{6}\,2}$$

$$= \frac{5}{6} - \frac{t}{6}$$

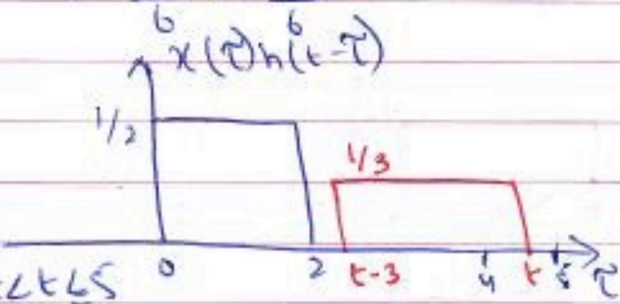


when ~~t~~ $3 < t \le 4$

$$y(t) = \int_{t-3}^{2} \left(\frac{1}{6}\right) d\tau$$

$$= \frac{1}{6}\left[\tau\right]_{t-3}^{2}$$

$$= \frac{5}{6} - \frac{t}{6}$$



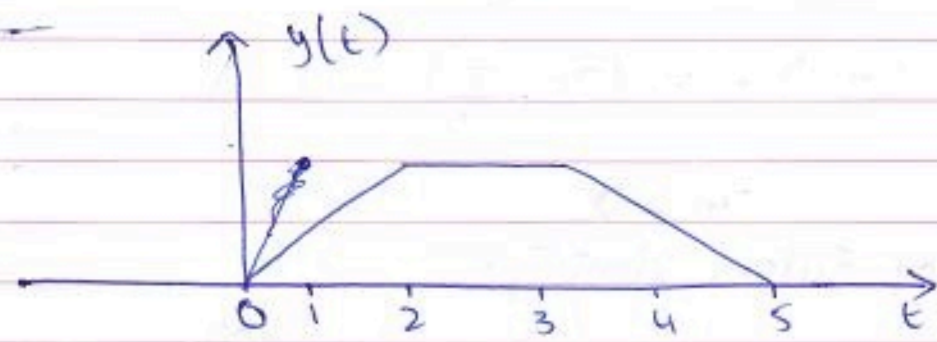
$4 < t \le 5$

$y(t) = 0$ as there is no overlapping

$\frac{2}{63}$ $\frac{15-6}{18}$

$\frac{5}{6} - \frac{2}{63}$

$$y(t) = \begin{cases} 0 & t<0 \\ t/6 & 0<t\le 1 \\ t/6 & 1<t\le 2 \\ \frac{5}{6}-\frac{t}{6} & 2<t\le 3 \\ \frac{5}{6}-\frac{t}{6} & 3<t\le 4 \\ 0 & t>4 \end{cases}$$



Example #2:-

$$x(t) = \cos(4t) + \sin(6t) \qquad T_0 = \pi$$

Complex exponential representation =?

Sol:-

$T_0 = \pi$

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{\pi} \Rightarrow 2$$

Thus,

$$x(t) = \cos(4t) + \sin(6t) = \sum_{k=-\infty}^{\infty} c_k e^{-j2kt}$$

* Using Euler's formula, we have

$$x(t) = \frac{1}{2}\left(e^{j4t} + e^{-4jt}\right) + \frac{1}{2j}\left(e^{j6t} - e^{-j6t}\right)$$

$$= -\frac{1}{2j} e^{-j6t} + \frac{1}{2} e^{-4jt} + \frac{1}{2} e^{j4t} + \frac{1}{2j} e^{j6t}$$

Thus, the complex Fourier coefficients for $\cos 4t + \sin 6t$ are:

$$c_{-3} = -\frac{1}{2j}, \quad c_{-2} = \frac{1}{2}, \quad c_2 = \frac{1}{2}, \quad c_3 = \frac{1}{2j}$$

and all other $c_k = 0$.

<u>Example #3:</u>

$$x(t) = \cos\left(2t + \frac{\pi}{4}\right)$$

<u>Sol:</u>

$$x(t) = \cos\left(2t + \frac{\pi}{4}\right)$$

$$\omega_0 = 2$$

Using Euler's identity,

$$x(t) = \frac{1}{2}\left[e^{j(2t+\pi/4)} + e^{-j(2t+\pi/4)}\right]$$

$$= \frac{1}{2} e^{-j2t} e^{-j\pi/4} + \frac{1}{2} e^{j2t} e^{j\pi/4}$$

The fourier series coefficients are,

$$c_1 = \frac{1}{2} e^{j\pi/4}, \quad c_{-1} = \frac{1}{2} e^{-j\pi/4}$$

$$c_k = 0.$$

EXAMPLE # 4:-

$$x[n] = \cos^2\left(\frac{\pi}{8}n\right), N_0 = 8$$

Sol:-

$$\omega_0 = \frac{2\pi}{N_0} = \frac{\cancel{2}\pi}{\cancel{8}4} \Rightarrow \frac{\pi}{4}$$

$$x[n] = \left(\frac{1}{2}e^{j(\pi/8)n} + \frac{1}{2}e^{-j(\pi/8)n}\right)^2$$

$$x[n] = \frac{1}{4}e^{j(\pi/4)n} + \frac{1}{2} + \frac{1}{4}e^{-j(\pi/4)n}$$

Thus, $c_0 = \frac{1}{2}$, $c_1 = \frac{1}{4}$, $c_{-1} = \frac{1}{4}$ and all other $c_k = 0$.

EXAMPLE # 5:-

$$X(e^{j\omega}) = \frac{1}{(1-ae^{-j\omega})^2} \quad |a|<1$$

Sol:-

we have, $a^n u[n] \leftrightarrow \frac{1}{1-ae^{-j\omega}} \quad |a|<1$

Now, $X(e^{j\omega}) = \frac{1}{(1-ae^{-j\omega})^2} = \left(\frac{1}{1-ae^{-j\omega}}\right)\left(\frac{1}{1-ae^{-j\omega}}\right)$

Thus, by the convolution theorem we get,

$$x[n] = a^n u[n] * a^n u[n] = \sum_{k=-\infty}^{\infty} a^k u[k] a^{n-k} u[n-k]$$

$$\cancel{= a^n} = \sum_{k=0}^{\infty} a^{k+n-k} = \sum_{k=0}^{\infty} a^n$$

$$x[n] = a^n \sum_{k=0}^{\infty} 1 \Rightarrow (n+1)a^n u[n]$$

Hence, $(n+1)a^n u[n] \leftrightarrow \frac{1}{(1-ae^{-j\omega})^2}$, $|a|<1$

EXAMPLE # 6:-

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{2}x[n-1] \rightarrow (1)$$

Sol:-

a) $H(e^{j\omega}) = ?$

Taking Fourier transform of equ (1)

$$Y(e^{j\omega}) - \frac{1}{2}e^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega}) + \frac{1}{2}e^{-j\omega}X(e^{j\omega})$$

$$Y(e^{j\omega})\left[1 - \frac{1}{2}e^{-j\omega}\right] = X(e^{j\omega})\left[1 + \frac{1}{2}e^{-j\omega}\right]$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1+\frac{1}{2}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$$

b) $h[n] = ?$

$$H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} + \frac{1}{2}\frac{e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}.$$

Taking the inverse Fourier transform,

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

EXAMPLE # 7:-

$$x[n] = \{5, 3, \underset{\uparrow}{-2}, 0, 4, -3\}$$

$X(z) = ?$

$ROC = ?$

Sol:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-2}^{3} x[n] z^{-n}$$

$$= x[-2]z^{2} + x[-1]z^{1} + x[0]z^{0} + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3}$$

$$= 5z^{2} + 3z - 2 + 4z^{-2} - 3z^{-3}.$$

For z not equal to zero or infinity, each term in $X(z)$ will be finite and consequently $X(z)$ will converge.

$X(z)$ includes both positive and negative powers of z, thus the ROC of $X(z)$ is $0 < |z| < \infty$.

Example # 8:

$$x[n] = \left(\frac{1}{2}\right)^{n} u[n] + \left(\frac{1}{3}\right)^{n} u[n]$$

$X(z) = ?$ pole-zero plot $= ?$

Sol:

From table of basic pairs,

$$\left(\frac{1}{2}\right)^{n} u[n] \Leftrightarrow \frac{1}{1-\frac{1}{2}z^{-1}} \text{ or } \frac{z}{z-\frac{1}{2}}, \quad |z| > \frac{1}{2}$$

$$\left(\frac{1}{3}\right)^{n} u[n] \Leftrightarrow \frac{1}{1-\frac{1}{3}z^{-1}} \text{ or } \frac{z}{z-\frac{1}{3}}, \quad |z| > \frac{1}{3}$$

$$X(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z-\frac{1}{3}}$$

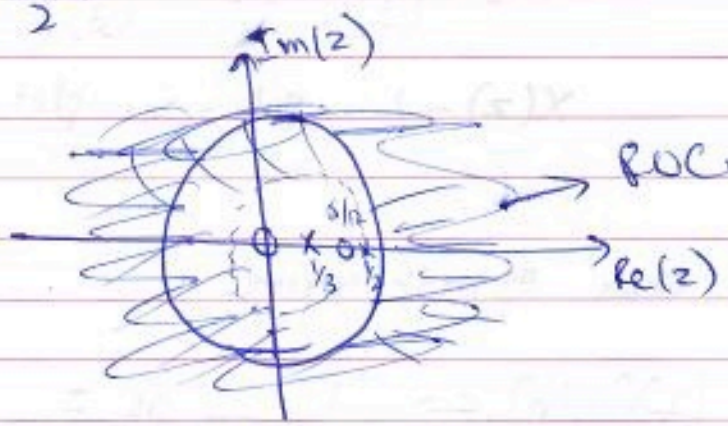
$$= \frac{z\left(z-\frac{1}{3}\right) + z\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$$= \frac{z^2-\frac{z}{3}+z^2-\frac{z}{2}}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$$= \frac{2z^2-\frac{5z}{6}}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)} = \frac{2z\left(z-\frac{5z}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$X(z)$ has two zeros at $z=0$ and $z=\frac{5}{12}$.

two poles at $z=\frac{1}{2}$ and $z=\frac{1}{3}$ and the ROC is $|z|>\frac{1}{2}$.



EXAMPLE #9:

$$X(z) = \frac{z}{z(z-1)(z-2)^2}, \quad |z|>2$$

Sol:

Using partial fraction expansion.

$$X(z) = \frac{\cancel{z}}{\cancel{z}(z-1)(z-2)^2} \Rightarrow$$

$$= \frac{1}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2} \rightarrow ①$$

$z=1$, $z=2$

putting $z=1$ equ ① and cross multiplication.

$$1 = A(z-2)^2 + B(z-1)(z-2) + C(z-1) \rightarrow ②$$

∴ $z=1$

$$1 = A(1-2)^2 + B\cancel{(1-1)}^{0}(1-2) + C\cancel{(1-1)}^{0}$$

$$1 = A(-1)^2$$

$$A \Rightarrow 1$$

Now $z=2$ in equ ②

$$1 = A\cancel{(2-2)}^{0\,2} + B\cancel{(2-1)}^{0}(2-2) + C(2-1)$$

$$1 = C(1)$$

$$C \Rightarrow 1$$

Now put values of A and C in equ ② and let $z=0$

$$1 = (1)(0-2)^2 + B(0-1)(0-2) + 1(0-1)$$

$$1 = (-2)^2 + B(-1)(-2) + 1(-1)$$

$$1 = 4 + 2B - 1$$

$$1 = +3 + 2B$$

$$2B = 1-3$$

$$2B = -2$$

$$B = \frac{-2}{2} \Rightarrow -1$$

putting all values of A, B & C in equ (1) gives

$$X(z) = \frac{1}{z-1} - \frac{1}{z-2} + \frac{1}{(z-2)^2} \qquad |z| > 2$$

Taking inverse z-transform.

$$x[n] = [1 - (2)^n + n(2)^{n-1}]u[n].$$