Signal & Systems

Lecture # 13

Z-Transform -II

4th January 18

Properties of ROC

Properties of ROC

- Property #1: The ROC is a ring or disk in the z-plane center at origin.
- Property #2: The Fourier transform of x|n| converges absolutely if and only if the ROC of the z-transform includes the unit circle.
- Property #3: The ROC contains no poles.
- Property #4: If x|n| is a finite impulse response (FIR), then the ROC is the entire z-plane.
- Property #5: If x|n| is a right-sided sequence then the ROC extends outwards from the outermost finite pole to infinity.
- Property #6: If x|n| is left sided then the ROC extends inward from the innermost nonzero pole to z=0.

Properties of ROC (cont.)

- Property #7: If $X(z)$ is rational, i.e., $X(z)=A(z) / B(z)$ where $A(z)$ and $B(z)$ are polynomials, and if $x[n]$ is right-sided, then the ROC is the region outside the outermost pole.

Inverse Z-transform

Inverse Z-Transform

- The inverse z-transform is used to derive x[n] from X(z).
- There are different methods with which we can derive x[n] if X(z) is given.
- By considering r fixed, inverse of z-transform can be obtained from inverse of Fourier transform. That is :

$$X\left(re^{j\omega}\right) = F\left\{x\left[n\right]r^{-n}\right\}$$

- For any value of r so that $z=re^{j\omega}$ is inside the ROC.
- Applying the inverse Fourier transform to both sides of above equation yields:

Inverse Z-Transform (cont.)

$$x[n]r^{-n} = F^{-1}\left\{X\left(re^{j\omega}\right)\right\}$$

$$or$$

$$x[n] = r^{n}F^{-1}\left\{X\left(re^{j\omega}\right)\right\}$$

- Using the inverse Fourier transform expression in above equation we have:

$$x[n] = r^{n}\frac{1}{2\pi}\int_{2\pi} X\left(re^{j\omega}\right)e^{j\omega n}\,d\omega$$

- Or moving the exponential factor r^n inside the integral and combining it with the term $e^{j\omega n}$:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X\left(re^{j\omega}\right)\left(re^{j\omega}\right)^{n}\,d\omega$$

Inverse Z-Transform (cont.)

- That is we can recover x[n] from its z-transform evaluated along a contour $z=re^{j\omega}$ in the ROC with r fixed and ω varying over a 2π interval.
- Consequently in terms of an integration in the z-plane above equation becomes:

$$x[n]=\frac{1}{2\pi j}\oint X(z)z^{n-1}\,dz$$

- Methods to obtain Inverse Z-transform:
  - If X(z) is rational, we can use expanding the rational algebraic into a linear combination of lower order terms and then one may use:
    - If ROC is out of pole $z = a_i$:

$$X(z)=\frac{A_i}{1-a_iz^{-1}}\rightarrow x[n]=A_ia_iu[n]$$

Inverse Z-Transform (cont.)

- If ROC is inside of $z = a_i$:

$$X(z) = \frac{A_i}{1 - a_i z^{-1}} \rightarrow x[n] = -A_i a_i u[-n-1]$$

- Do not forget to consider ROC in obtaining inverse of ZT.

- If X(z) is non-rational, use Power series expansion of X(z), then apply $\delta[n+n_o] \leftrightarrow z^{no}$
- If X(z) is rational, power series can be obtained by long division.
- If X(z) is a rational function of z, i.e., a ratio of polynomials, we can also use partial fraction expansion to express X(z) as a sum of simple terms for which the inverse transform may be recognized by inspection.
- The ROC plays a critical role in this process.

Example #1

- Consider the z-transform:

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)}, |z| > \frac{1}{3}$$

Example #2

- Consider the signal:

$$X(z) = 4z^{2} + 2 + 3z^{-1}, 0 < |z| < \infty$$

Analysis & Characterization of LTI Systems

LTI Systems Using z-Transforms

- The z-transform plays a particularly important role in the analysis and representation of discrete time LTI system.
- From the convolution property: $Y(z) = H(z) X(z)$.
- Where $X(z)$, $Y(z)$ and $H(z)$ are the z-transforms of the system input, output and impulse response respectively.
- $H(z)$ is referred to as the system function or transfer function of the system.
- For z-evaluated on the unit circle (i.e., $z = e^{j\omega}$), $H(z)$ reduces to the frequency response of the system, provided that the unit circle is in the ROC for $H(z)$.
- We know that if the input to an LTI system is the complex exponential $x[n] = z^n$, then the output will be $H(z) z^n$.
- That is z^n is an Eigen-function of the system with eigenvalue given by $H(z)$, the z-transform of the impulse response.

LTI Systems Using z-Transforms (cont.)

- Many properties of a system can be tied directly to characteristics of the poles, zeros, and region of convergence of the system function.
- Lets discuss few of the system properties of z-transform, that is:
  - Causality
  - Stability

Causality

- A discrete-time LTI system is causal if and only if ROC is the exterior of a circle (including ∞).
- Proof:
- A system is causal if and only if: h[n] = 0, n<0.
- Therefore, h[n] must be right sided. Property 5 implies that ROC is outside a circle.
- Also, by the definition that:

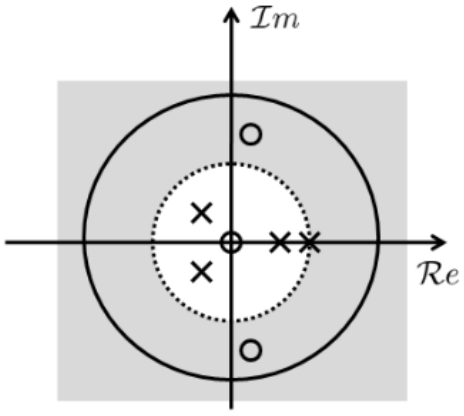
$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

- Where there is no positive powers of z, H(z) converges also when z➔∞.
- |z|>1 when z➔∞.

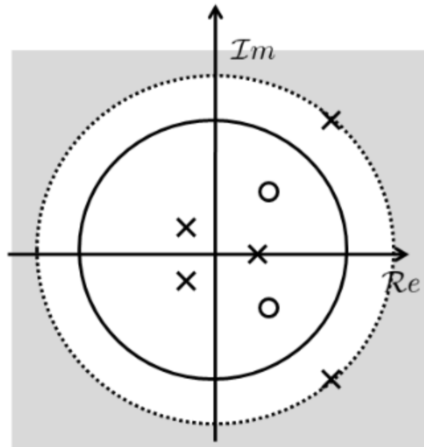
Stability

- A discrete time LTI system is stable if and only if ROC of $H(z)$ includes the unit circle.
- Proof: A system is stable if and only if $h[n]$ is absolutely summable, if and only if DTFT of $h[n]$ exists. Consequently by property 2, ROC of $H(z)$ must include the unit circle.
- A causal discrete-time LTI system is stable if and only if all of its poles are inside the unit circle.

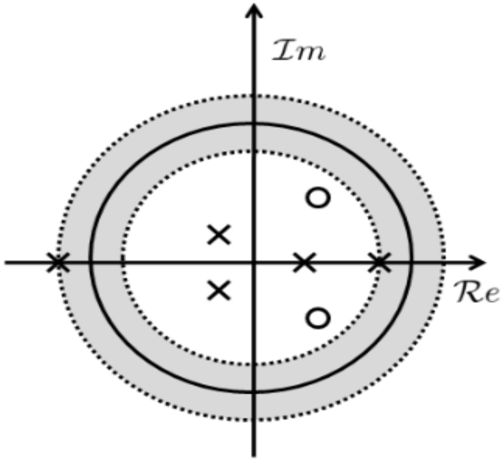
Examples



Causal, Stable



Causal, Unstable



Not causal, Stable

Example #3

- Consider a system with system function:

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}, |z| > 2$$

- Whether the system is Causal or not?
- Find h[n]=?

LTI Systems Characterized by LCCDE

- For systems characterized by liner constant-coefficient difference equations, the properties of the z-transform provide a particularly convenient procedure for obtaining the system function, function response or time domain response of the system.

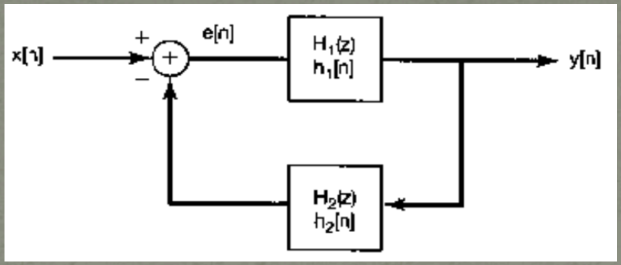
Example #4

- Consider an LTI system for which the input x[n] and the output y[n] satisfy the linear constant coefficient difference equation:

$$y[n]-\frac{1}{2}y[n-1]=x[n]+\frac{1}{3}x[n-1]$$

System Functions for Interconnections of LTI Systems

- The system function for the cascade of two discrete-time LTI systems is the product of the system functions for the individual systems in the cascade.
- Feedback interconnection of two systems is shown below:



- It is involved to determine the difference equation or impulse response for the overall system working directly in the time domain.
- However with the systems and sequences expressed in terms of their z-transforms, the analysis involves only algebraic equations.

System Functions for Interconnections of LTI Systems (cont.)

$$Y(z) = Y_1(z) = X_2(z)$$

$$X_1(z) = X(z) - Y_2(z) = X(z) - H_2(z)Y(z)$$

$$Y(z) = H_1(z)X_1(z) = H_1(z)\left[X(z) - H_2(z)Y(z)\right]$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{H_1(z)}{1 + H_2(z)H_1(z)}$$

- ROC is determined based on roots of $1+H_2(z)H_1(z)$.

Block Diagram Representation for Causal LTI System

- Causal LTI systems can be described by difference equations using block diagram involving three basic operations:
  - Addition
  - Multiplication by a coefficient
  - A unit delay

Example #5

- Consider the causal LTI system with system function:

$$H(z)=\frac{1-2z^{-1}}{1-\frac{1}{4}z^{-1}}=\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)\left(1-2z^{-1}\right)$$

The End