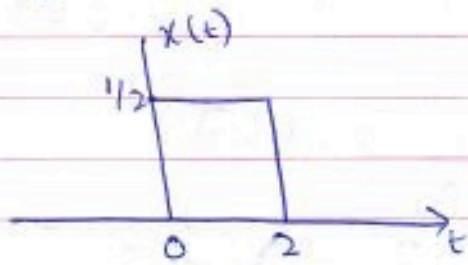


LECTURE #14

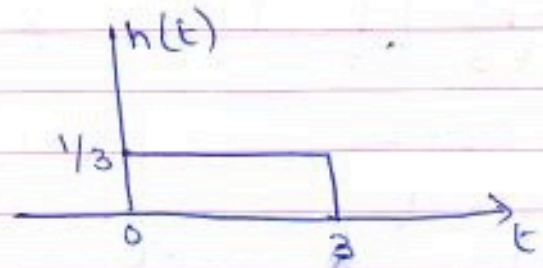
: SOLVED EXAMPLE

day / date: Fri / 11-01-18

EXAMPLE #1:-

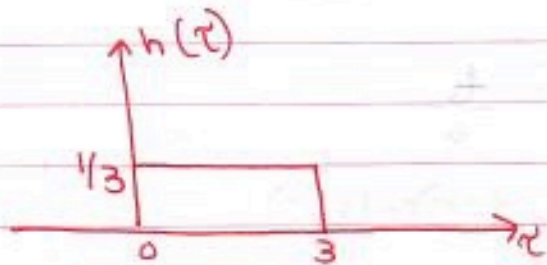
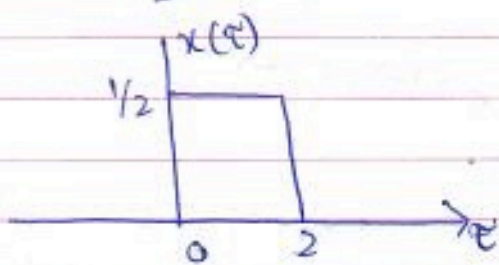


*

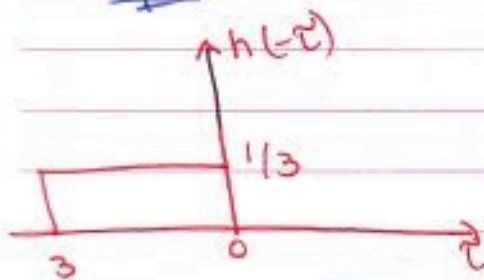


Sol:-

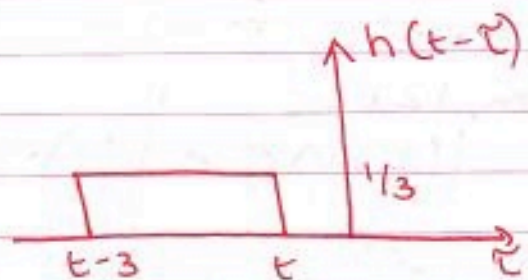
Step #1:-



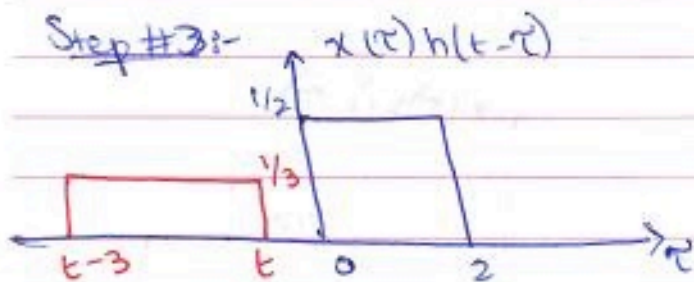
Step #2:-



→

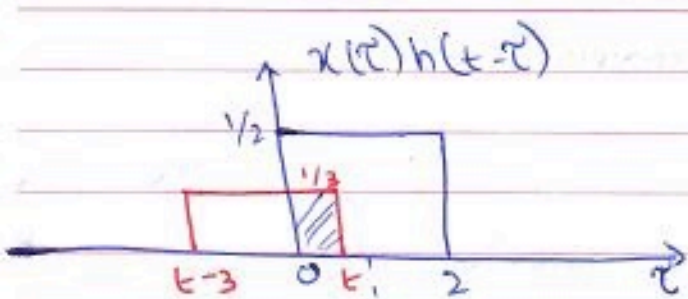


Step #3:-



$t < 0$

$y(t) = 0$ as there is no overlapping.

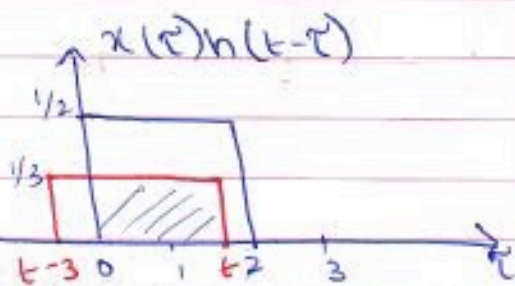


when $0 < t \leq 1$

$$y(t) = \int_0^t \left(\frac{1}{2} \times \frac{1}{3}\right) d\tau$$

$$= \frac{1}{6} \int_0^t d\tau = \frac{1}{6} [t - 0]$$

$$y(t) = \frac{t}{6}$$



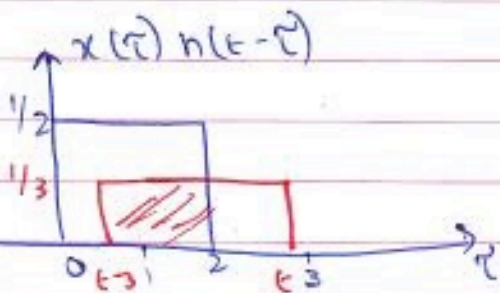
when, $1 < t \leq 2$

$$y(t) = \int \left(\frac{1}{2} \times \frac{1}{3}\right) d\tau + \int \left(\frac{1}{6}\right) d\tau$$

$$= \frac{1}{6} [\tau]_0^1 + \frac{1}{6} [\tau]_1^t$$

$$= \frac{1}{6} [1 - 0] + \frac{1}{6} [t - 1]$$

$$= \frac{1}{6} + \frac{1}{6}t - \frac{1}{6} \Rightarrow \frac{t}{6}$$



when $2 < t \leq 3$

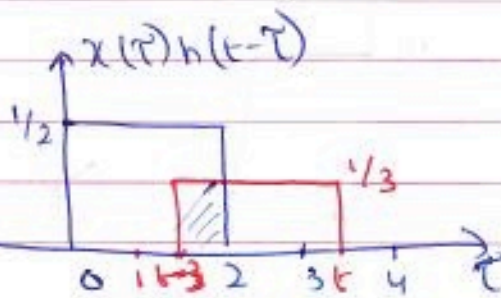
$$y(t) = \int_{t-3}^2 \left(\frac{1}{6}\right) d\tau$$

$$= \frac{1}{6} [\tau]_{t-3}^2 = \frac{1}{6} [2 - (t-3)]$$

$$= \frac{1}{6} [2 - t + 3]$$

$$y(t) = \frac{2}{6} - \frac{t}{6} + \frac{3}{6}$$

$$= \frac{5}{6} - \frac{t}{6}$$



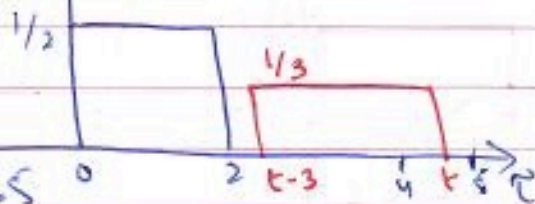
when $3 < t \leq 4$

$$y(t) = \int_{t-3}^2 \left(\frac{1}{6}\right) d\tau$$

$$= \frac{1}{6} [\tau]_{t-3}^2$$

$$= \frac{2}{6} - \frac{t}{6}$$

$$= \frac{2}{6} - \frac{t}{6}$$



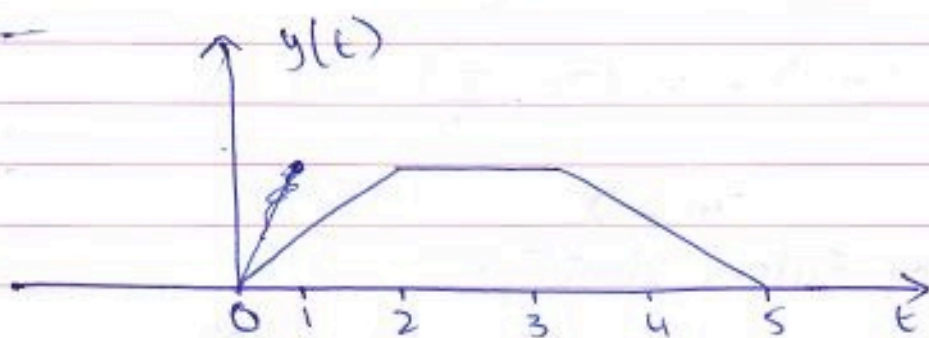
$t > 4$

$y(t) = 0$ as there is no overlapping

$$y(t) = \begin{cases} 0 & t < 0 \\ t/6 & 0 < t \leq 1 \\ t/6 & 1 < t \leq 2 \\ \frac{5}{6} - \frac{t}{6} & 2 < t \leq 3 \\ \frac{5}{6} - \frac{t}{6} & 3 < t \leq 4 \\ 0 & t > 4 \end{cases}$$

$$\frac{2}{6} = \frac{15-6}{18}$$

$$\frac{5}{6} - \frac{2}{6}$$



EXAMPLE #2:-

$$x(t) = \cos(4t) + \sin(6t) \quad T_0 = \pi$$

Complex exponential representation = ?

Soln

$$T_0 = \pi$$

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{\pi} \Rightarrow 2$$

Thus,

$$x(t) = \cos(4t) + \sin(6t) = \sum_{k=-\infty}^{\infty} c_k e^{-j2kt}$$

Using Euler's formula, we have

$$x(t) = \frac{1}{2} (e^{j4t} + e^{-j4t}) + \frac{1}{2j} (e^{j6t} - e^{-j6t})$$

$$= \frac{1}{2j} e^{-j6t} + \frac{1}{2} e^{j4t} + \frac{1}{2} e^{j4t} + \frac{1}{2j} e^{j6t}$$



Thus, the complex Fourier coefficients for $\cos 4t + \sin 6t$ are:

$$C_{-3} = -\frac{1}{2j}, \quad C_{-2} = \frac{1}{2}, \quad C_2 = \frac{1}{2}, \quad C_3 = \frac{1}{2j}$$

and all other $C_k = 0$.

EXAMPLE #38

$$x(t) = \cos\left(2t + \frac{\pi}{4}\right)$$

Soln-

$$x(t) = \cos\left(2t + \frac{\pi}{4}\right)$$

$$\omega_0 = 2$$

Using Euler's identity,

$$\begin{aligned} x(t) &= \frac{1}{2} \left[e^{j(2t + \pi/4)} + e^{-j(2t + \pi/4)} \right] \\ &= \frac{1}{2} e^{j2t} e^{j\pi/4} + \frac{1}{2} e^{-j2t} e^{-j\pi/4} \end{aligned}$$

The Fourier series coefficients are,

$$C_1 = \frac{1}{2} e^{j\pi/4}, \quad C_{-1} = \frac{1}{2} e^{-j\pi/4}$$

$$C_k = 0.$$

EXAMPLE # 4:

$$x[n] = \cos^2\left(\frac{\pi}{8}n\right), N_0 = 8$$

Soln-

$$\omega_0 = \frac{2\pi}{N_0} = \frac{2\pi}{8} \Rightarrow \frac{\pi}{4}$$

$$x[n] = \left(\frac{1}{2}e^{j(\pi/8)n} + \frac{1}{2}e^{-j(\pi/8)n}\right)^2$$

$$x[n] = \frac{1}{4}e^{j(\pi/4)n} + \frac{1}{2} + \frac{1}{4}e^{-j(\pi/4)n}$$

Thus, $c_0 = \frac{1}{2}$, $c_1 = \frac{1}{4}$, $c_{-1} = \frac{1}{4}$ and all other $c_k = 0$.

EXAMPLE # 5:

$$X(e^{j\omega}) = \frac{1}{(1 - ae^{-j\omega})^2}, |a| < 1$$

Soln-

we have, $a^n u[n] \leftrightarrow \frac{1}{1 - ae^{-j\omega}}, |a| < 1$

Now, $X(e^{j\omega}) = \frac{1}{(1 - ae^{-j\omega})^2} = \left(\frac{1}{1 - ae^{-j\omega}}\right) \left(\frac{1}{1 - ae^{-j\omega}}\right)$

Thus, by the convolution theorem we get,

$$x[n] = a^n u[n] * a^n u[n] = \sum_{k=-\infty}^{\infty} a^k u[k] a^{n-k} u[n-k]$$

$$\Rightarrow x[n] = \sum_{k=0}^{\infty} a^{k+n-k} = \sum_{k=0}^{\infty} a^n$$

$$x[n] = a^n \sum_{k=0}^{\infty} 1 \Rightarrow (n+1) a^n u[n]$$

Hence, $(n+1) a^n u[n] \leftrightarrow \frac{1}{(1 - ae^{-j\omega})^2}, |a| < 1$

EXAMPLE # 68

$$y[n] - \frac{1}{2} y[n-1] = x[n] + \frac{1}{2} x[n-1] \rightarrow \textcircled{1}$$

Soln

a) $H(e^{j\omega}) = ?$

Taking Fourier transform of equ $\textcircled{1}$

$$Y(e^{j\omega}) - \frac{1}{2} e^{-j\omega} Y(e^{j\omega}) = X(e^{j\omega}) + \frac{1}{2} e^{-j\omega} X(e^{j\omega})$$

$$Y(e^{j\omega}) \left[1 - \frac{1}{2} e^{-j\omega} \right] = X(e^{j\omega}) \left[1 + \frac{1}{2} e^{-j\omega} \right]$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 + \frac{1}{2} e^{-j\omega}}{1 - \frac{1}{2} e^{-j\omega}}$$

b) $h[n] = ?$

$$H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}} + \frac{\frac{1}{2} e^{-j\omega}}{1 - \frac{1}{2} e^{-j\omega}}$$

Taking the inverse Fourier transform,

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{2} \left(\frac{1}{2}\right)^{n-1} u[n-1]$$

EXAMPLE # 70

$$x[n] = \{5, 3, -2, 0, 4, -3\}$$

$$X(z) = ?$$

$$\text{ROC} = ?$$

Solve

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \sum_{n=-2}^3 x[n]z^{-n}$$

$$= x[-2]z^2 + x[-1]z^1 + x[0]z^0 + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3}$$

$$= 5z^2 + 3z - 2 + 4z^{-2} - 3z^{-3}$$

For z not equal to zero or infinity, each term in $X(z)$ will be finite and consequently $X(z)$ will converge.

$X(z)$ includes both positive and negative powers of z , thus the ROC of $X(z)$ is $0 < |z| < \infty$.

EXAMPLE # 88

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$$

$$X(z) = ? \quad \text{pole-zero plot} = ?$$

Solve

From table of basic pairs,

$$\left(\frac{1}{2}\right)^n u[n] \leftrightarrow \frac{1}{1 - \frac{1}{2}z^{-1}} \leftrightarrow \frac{z}{z - \frac{1}{2}}, \quad |z| > \frac{1}{2}$$

$$\left(\frac{1}{3}\right)^n u[n] \leftrightarrow \frac{1}{1 - \frac{1}{3}z^{-1}} \leftrightarrow \frac{z}{z - \frac{1}{3}}, \quad |z| > \frac{1}{3}$$

$$X(z) = \frac{z}{z - \frac{1}{2}} + \frac{z}{z - \frac{1}{3}}$$

$$= \frac{z(z - \frac{1}{3}) + z(z - \frac{1}{2})}{(z - \frac{1}{2})(z - \frac{1}{3})}$$

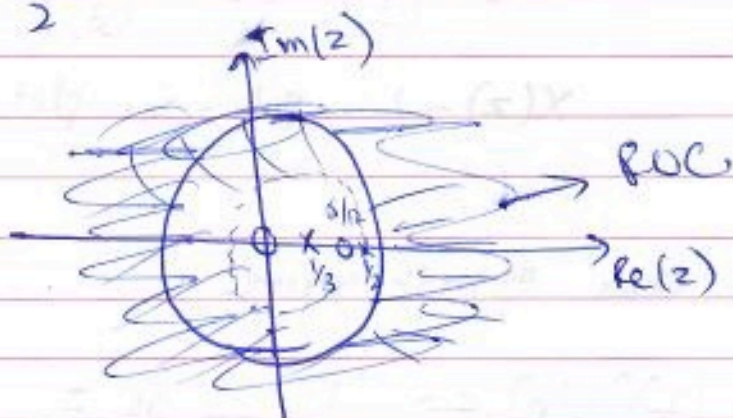
$$= \frac{z^2 - \frac{z}{3} + z^2 - \frac{z}{2}}{(z - \frac{1}{2})(z - \frac{1}{3})}$$

$$= \frac{2z^2 - \frac{5z}{6}}{(z - \frac{1}{2})(z - \frac{1}{3})} = \frac{2z(z - \frac{5z}{12})}{(z - \frac{1}{2})(z - \frac{1}{3})}$$

$X(z)$ has two zeros at $z=0$ and $z = \frac{5}{12}$.

two poles at $z = \frac{1}{2}$ and $z = \frac{1}{3}$ and the ROC

is $|z| > \frac{1}{2}$.



EXAMPLE #9

$$X(z) = \frac{z}{z(z-1)(z-2)^2}, \quad |z| > 2$$

Soln

Using partial fraction expansion.

$$\begin{aligned} X(z) &= \frac{z}{z(z-1)(z-2)^2} \rightarrow \\ &= \frac{1}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2} \rightarrow (1) \end{aligned}$$

$$z=1, \quad z=2$$

putting $z=1$ equ (1) and cross multiplication,

$$1 = A(z-2)^2 + B(z-1)(z-2) + C(z-1) \rightarrow (2)$$

∴ $z=1$

$$1 = A(1-2)^2 + B(1-1)(1-2) + C(1-1)$$

$$1 = A(-1)^2$$

$$A \Rightarrow 1$$

Now $z=2$ in equ (2)

$$1 = A(2-2)^2 + B(2-1)(2-2) + C(2-1)$$

$$1 = C(1)$$

$$C \Rightarrow 1$$

~~Now let $z=0$~~ Now put values of A and C in equ (2) and let $z=0$

$$1 = (1)(0-2)^2 + B(0-1)(0-2) + 1(0-1)$$

$$1 = (-2)^2 + B(-1)(-2) + 1(-1)$$

$$1 = 4 + 2B - 1$$

$$1 = +3 + 2B$$

$$2B = 1 - 3$$

$$2B = -2$$

$$B = \frac{-2}{2} \Rightarrow -1$$

putting all values of A, B & C in equ (1) gives

$$X(z) = \frac{1}{z-1} - \frac{1}{z-2} + \frac{1}{(z-2)^2} \quad |z| > 2$$

Taking inverse z-transform.

$$x[n] = [1 - (2)^n + n(2)^{n-1}] u[n].$$