# Signal & Systems

Lecture # 9

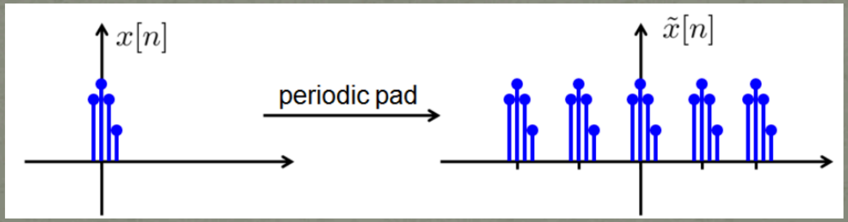
Discrete Time Fourier Transform-I

1st January 19

# Discrete Time Fourier Transform (DTFT)

# Development of the DTFT

- In deriving discrete-time Fourier Transform we have three key steps:
- Step#1:
  - Consider an aperiodic discrete-time signal x[n]. We pad x[n] to construct a periodic signal x'[n].




- Step#2:
  - Since x'[n] is periodic, by discrete-time Fourier series we have:

$$x'[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

# Development of the DTFT (cont.)

- Where $a_k$ is:

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x'[n]e^{-jk(2\pi/N)n}$$

- Here, $\omega_o=2\pi/N$.
- Now note that x'[n] is a periodic signal with period N and the non-zero entries of x'[n] in a period are the same as the non-zero entries of x[n].
- Therefore, it holds that:

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x'[n]e^{-jk(2\pi/N)n}$$

$$= \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-jk(2\pi/N)n}$$

# Development of the DTFT (cont.)

- If we define: $$X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}$$
- Then: $$a_k=\frac{1}{N}\sum_{n=-\infty}^{\infty}x[n]e^{-jk(2\pi/N)n}=\frac{1}{N}X\left(e^{jk\omega_0}\right)$$
- Step#3:
  - Putting above equation in discrete-time Fourier series equation, we have: $$x'[n]=\sum_{k=\langle N\rangle}a_k e^{jk\omega_0 n}$$

$$=\sum_{k=\langle N\rangle}\left[\frac{1}{N}X\left(e^{jk\omega_0}\right)\right]e^{jk\omega_0 n}$$

$$=\frac{1}{2\pi}\sum_{k=\langle N\rangle}X\left(e^{jk\omega_0}\right)e^{jk\omega_0 n}\omega_0,\quad \omega_0=\frac{2\pi}{N}$$

# Development of the DTFT (cont.)

- As N→∞, $\omega_0$→0, so the area becomes infinitesimal small and sum becomes integration and x'[n]=x[n],so above equation becomes,

$$x'[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X\left(e^{jk\omega_0}\right) e^{jk\omega_0 n} \omega_0 \rightarrow \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} \, d\omega$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} \, d\omega$$

- Hence, the Discrete time Fourier transform pair:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

# Development of the DTFT (cont.)

- The first equation is referred to as synthesis equation and second one as analysis equation.
- $X(e^{j\omega})$ is referred to as the spectrum of x[n].

# Example #1

- Consider the signal:

$$x[n] = a^n u[n], \quad |a| < 1$$

- Solution:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n u[n]e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n = \frac{1}{1-ae^{-j\omega}}$$

# Example #2

- Consider the signal: $x[n] = a^{|n|}, \quad |a| < 1$

- Solution:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} a^{|n|} u[n] e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

- Let m=-n in the first summation we obtain,

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} a^{|n|} u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} a^{n} e^{-j\omega n} + \sum_{m=1}^{\infty} a^{m} e^{j\omega m}$$

$$= \sum_{n=0}^{\infty} \left(a e^{-j\omega}\right)^{n} + \sum_{m=1}^{\infty} \left(a e^{j\omega}\right)^{m}$$

# Example #2 (cont.)

- Both of these summations are infinite geometric series that we can evaluate in closed form, yielding:

$$X\left(e^{j\omega}\right) = \frac{1}{1-ae^{-j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}}$$

$$= \frac{1-a^2}{1-2a\cos\omega + a^2}$$

# The Fourier Transform of Periodic Signals

# Periodic Signals

- For a periodic discrete-time signal: $x[n] = e^{j\omega_0 n}$
- The discrete-time Fourier transform must be periodic in ω with period 2π.
- Then the Fourier transform of x[n] should have impulses at $\omega_0$ , $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$, and so on.
- In fact, the Fourier transform of x[n] is the impulse train:

$$X\left(e^{j\omega}\right) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

- Now consider a periodic sequence x[n] with period N and with the Fourier series representation:

# Periodic Signals (cont.)

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

- In this case, the Fourier transform is:

$$X\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

- So that the Fourier transform of a periodic signal can directly constructed from its Fourier coefficients.

# Example #3

- Consider the periodic signal:

$$x[n] = \cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}, \quad where \quad \omega_0 = \frac{2\pi}{5}$$

- Solution:
  - From the equation of periodicity we can write:

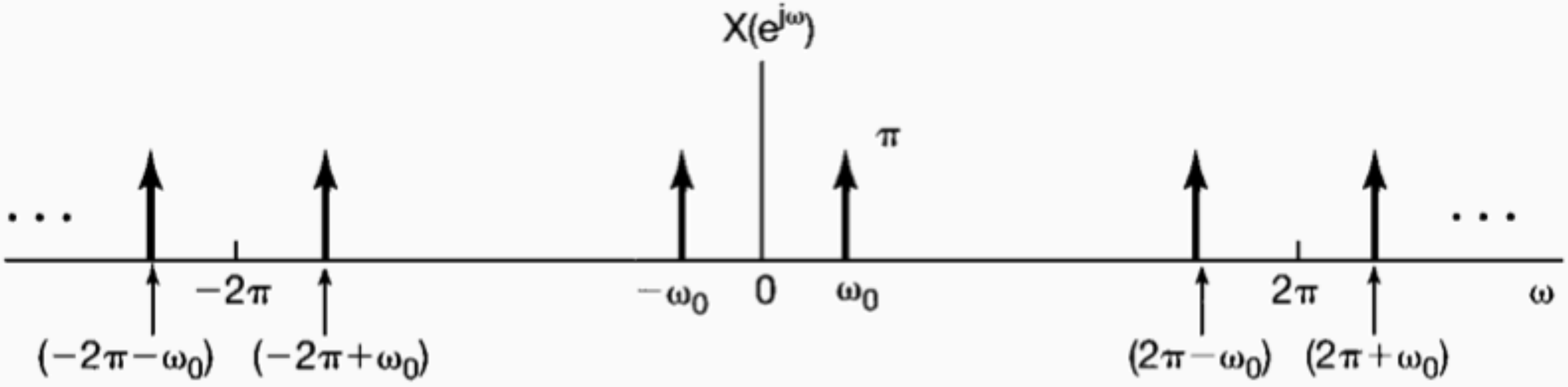
$$X\left(e^{j\omega}\right) = \sum_{l=-\infty}^{\infty} \pi\delta(\omega - \frac{2\pi}{5} - 2\pi l) + \sum_{l=-\infty}^{\infty} \pi\delta(\omega + \frac{2\pi}{5} - 2\pi l)$$

  - That is,

$$X\left(e^{j\omega}\right) = \pi\delta\left(\omega - \frac{2\pi}{5}\right) + \pi\delta\left(\omega + \frac{2\pi}{5}\right), \quad -\pi \le \omega < \pi$$

  - $X(e^{j\omega})$ repeats periodically with a period of $2\pi$, as shown below:

# Example #3 (cont.)




Discrete-time Fourier transform of $x[n] = \cos \omega_0 n$.

# Properties of DT Fourier Transform

# Periodicity

- The discrete-time Fourier transform is always periodic in ω with period 2π, i.e.,

$$X\left(e^{j(\omega+2\pi)}\right)=X\left(e^{j\omega}\right)$$

# Linearity

- If:

$$x_1[n] \leftrightarrow X_1\left(e^{j\omega}\right)$$

$$And$$

$$x_2[n] \leftrightarrow X_2\left(e^{j\omega}\right)$$

- Then:

$$ax_1[n] + bx_2[n] \overset{F}{\leftrightarrow} aX_1\left(e^{j\omega}\right) + bX_2\left(e^{j\omega}\right)$$

# Time Shifting & Frequency Shifting

- If:

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

- Then:

$$x[n-n_0] \overset{F}{\leftrightarrow} e^{-j\omega_0 n} X\left(e^{j\omega}\right)$$

$$and$$

$$e^{j\omega_0 n} x[n] \overset{F}{\leftrightarrow} X\left(e^{j(\omega-\omega_0)}\right)$$

# Conjugation & Conjugate Symmetry

- If: $x[n] \leftrightarrow X\left(e^{j\omega}\right)$
- Then: $x^*[n] \overset{F}{\leftrightarrow} X^*\left(e^{-j\omega}\right)$
- If x[n] is real valued, its transform $X(e^{j\omega})$ is conjugate symmetric. That is: $$X\left(e^{j\omega}\right) = X^*\left(e^{-j\omega}\right)$$
- From this, it follows that $\mathrm{Re}\left\{X\left(e^{j\omega}\right)\right\}$ is an even function of ω and $\mathrm{Im}\left\{X\left(e^{j\omega}\right)\right\}$ is an odd function of ω.
- Similarly the magnitude of $X(e^{j\omega})$ is an even function and the phase angle is an odd function.

# Conjugation & Conjugate Symmetry

- Furthermore,

$$Ev\left\{x[n]\right\} \overset{F}{\leftrightarrow} \mathrm{Re}\left\{X\left(e^{j\omega}\right)\right\}$$

*and*

$$Od\left\{x[n]\right\} \overset{F}{\leftrightarrow} j\,\mathrm{Im}\left\{X\left(e^{j\omega}\right)\right\}$$

# The End