Name: ______________________________ **Regd. No.** ______________

Course Title: Signal & Systems **Course Code:** ETSS-314

Solution

FINAL TERM EXAMINATION – Spring 2018

Program: B-Tech (Electrical)

SECTION-II: 40 MARKS

Time Allowed: 2hr 10 min

Attempt all questions. Marks are mentioned against the questions.   **Note: Please attach the question paper at the end of the answer sheet.**

Q2.

a) Find the Fourier Series Coefficients of the following signal:

$$x[n] = 1 + 3\cos\frac{2\pi n}{N}$$

(05 Marks)

Solution:

$$x[n] = 1 + 3\cos\frac{2\pi n}{N}$$

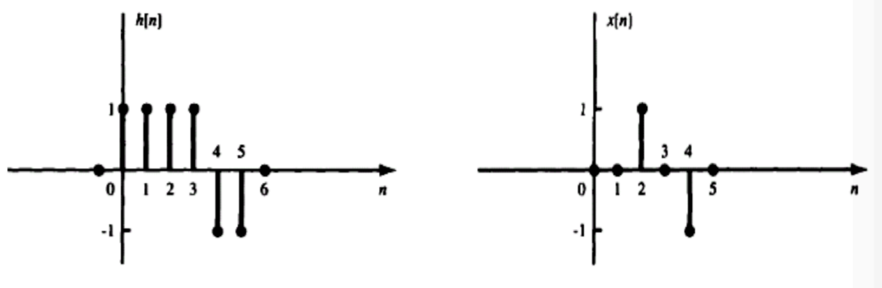
$$x[n] = 1 + \frac{3}{2}\left(e^{j\frac{2\pi n}{N}} + e^{-j\frac{2\pi n}{N}}\right)$$

The Fourier series coefficients are:

$$a_0 = 1, a_2 = \frac{3}{2}, a_{-2} = \frac{3}{2}$$

---

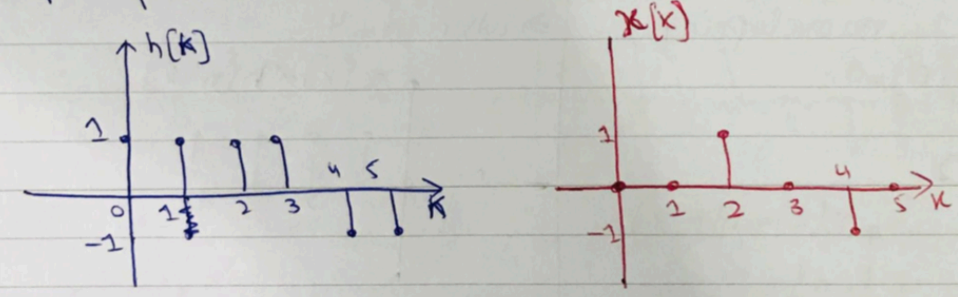
b) Evaluate $y[n] = x[n] * h[n]$. Using Graphical Method.



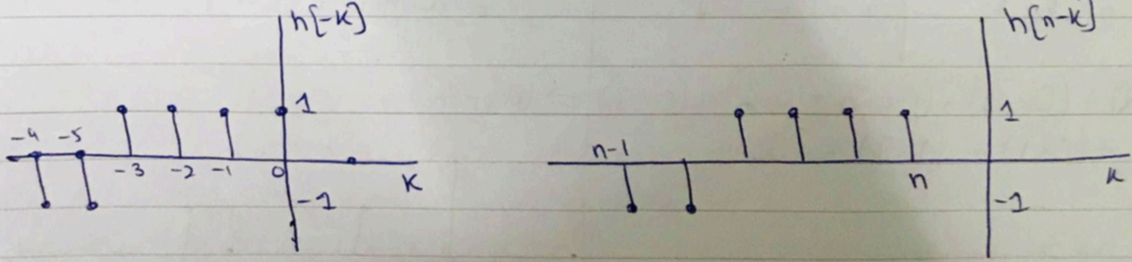
(08 Marks)

Solution:

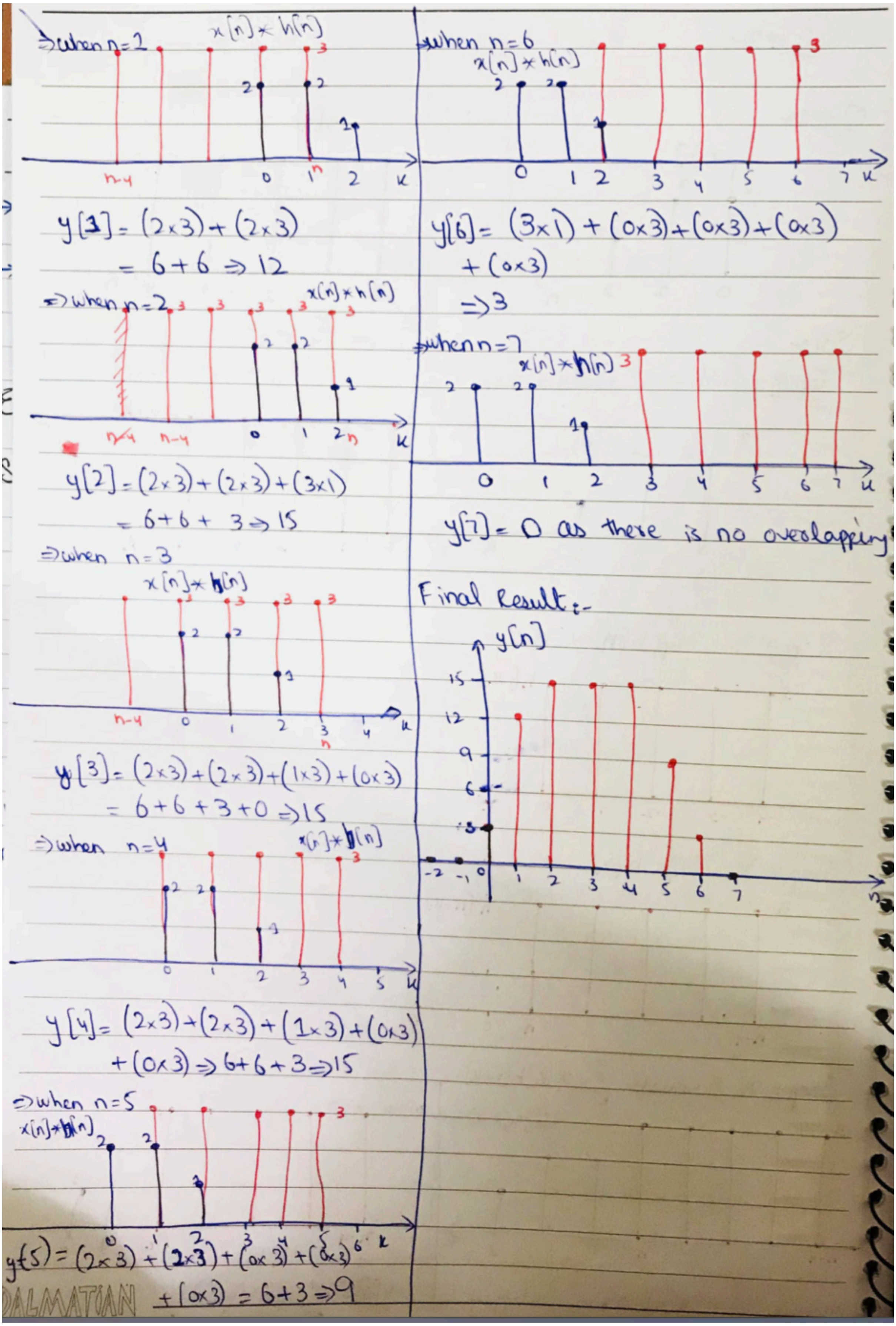
Step1: replace $n \to k$



Step 2: Flip and shift h[k].



Step3: Now slide h[n-k] on x[k] and convolve



Q3.

a) Compute the Discrete-Time Fourier Transform of the following signals:

i. $x[n] = \left(\frac{1}{4}\right)^n u[n]$

ii. $x[n] = 2\left(\frac{3}{4}\right)^n u[n]$

(08 Marks)

Solution:

i. $x[n] = \left(\frac{1}{4}\right)^n u[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left(\left(\frac{1}{4}\right)^n u[n]\right) e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{4}e^{-j\omega}\right)^n$$

Applying the transform pair: $(a)^n u[n] \overset{DTFT}{\longleftrightarrow} \frac{1}{1-ae^{-j\omega}}$

$$X(e^{j\omega}) \Rightarrow \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

---

ii. $x[n] = 2\left(\frac{3}{4}\right)^n u[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left(2\left(\frac{3}{4}\right)^n u[n]\right) e^{-j\omega n}$$

$$X(e^{j\omega}) = 2\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n e^{-j\omega n} = 2\sum_{n=0}^{\infty} \left(\frac{3}{4}e^{-j\omega}\right)^n$$

Applying the transform pair: $(a)^n u[n] \overset{DTFT}{\longleftrightarrow} \frac{1}{1-ae^{-j\omega}}$

$$X(e^{j\omega}) = 2\left[\frac{1}{1-\frac{3}{4}e^{-j\omega}}\right] \Rightarrow \frac{2}{1-\frac{3}{4}e^{-j\omega}}$$

---

b) Consider the linear constant coefficient difference equation:

$$y[n] - \frac{1}{4}y[n-1] = x[n]$$

Which describes a linear, time-invariant system initially at rest. Determine $H(e^{j\omega})$.

(06 Marks)

Solution:

The difference equation $y[n] - \frac{1}{4}y[n-1] = x[n]$, which is initially at rest, has a system transfer function that can be obtained by taking the Fourier Transform of both sides of the equation. This yields:

$$Y(e^{j\omega}) - \frac{1}{4}e^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega})$$

$$Y(e^{j\omega})\left(1 - \frac{1}{4}e^{-j\omega}\right) = X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \Rightarrow \frac{1}{1 - \frac{1}{4}e^{-j\omega}}$$

Q4.

a) Determine the z-transform (including the ROC) of the following sequences. Also sketch the pole-zero plots and indicate the ROC on your sketch:

$$\left(\frac{1}{3}\right)^n u[n]$$

(05 Marks)

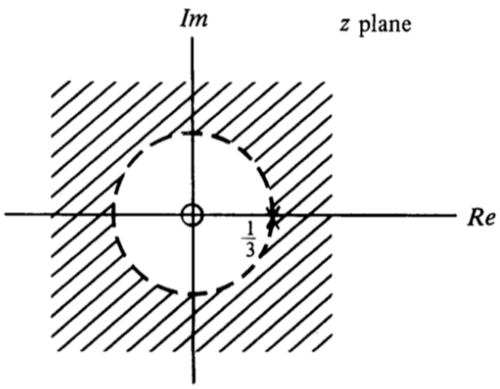
<u>**Solution:**</u>

$$\left(\frac{1}{3}\right)^n u[n] \overset{z}{\leftrightarrow} \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty}(3z)^{-n} = \frac{1}{1 - \frac{1}{3}z^{-1}} = \frac{z}{z - \frac{1}{3}}$$

Therefore, there is a zero at z = 0 and a pole at z = 1/3, and the ROC is:

$$\left|\frac{1}{3z}\right| < 1 \quad or \quad |z| > \frac{1}{3}$$



---

b) Using partial fraction expansion and the fact that:

$$(a)^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}} \;,\; |z| > |a|$$

Find the inverse z-transform of:

$$X(z) = \frac{1 - \frac{1}{3}z^{-1}}{(1 - z^{-1})(1 + 2z^{-1})} \;,\; |z| > 2$$

(08 Marks)

Solution:

Using partial fraction expansion:

$$X(z) = \frac{1 - \frac{1}{3}z^{-1}}{(1 - z^{-1})(1 + 2z^{-1})} = \frac{A}{(1 - z^{-1})} + \frac{B}{(1 + 2z^{-1})} \rightarrow eq(1)$$

Cross multiplication gives:

$$1 - \frac{1}{3}z^{-1} = A(1 + 2z^{-1}) + B(1 - z^{-1})$$

Putting $z^{-1} = 1$, gives:

$$1 - \frac{1}{3}(1) = A\big(1 + 2(1)\big) + B(1 - 1)$$

$$\frac{3 - 1}{3} = A(1 + 2) + B(0)$$

$$\frac{2}{3} = A(3) \Rightarrow A = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$$

Putting $z^{-1} = -\frac{1}{2}$, gives:

$$1 - \frac{1}{3}\left(-\frac{1}{2}\right) = A\left(1 + 2\left(-\frac{1}{2}\right)\right) + B\left(1 - \left(-\frac{1}{2}\right)\right)$$

$$1 + \frac{1}{6} = A(0) + B\left(1 + \frac{1}{2}\right)$$

$$\frac{6 + 1}{6} = B\left(\frac{2 + 1}{2}\right)$$

$$\frac{7}{6} = B\left(\frac{3}{2}\right) \Rightarrow A = \frac{7}{6} \times \frac{2}{3} = \frac{7}{9}$$

Putting values of A and B in eq(1) gives:

$$X(z) = \frac{\frac{2}{9}}{(1 - z^{-1})} + \frac{\frac{7}{9}}{(1 + 2z^{-1})}$$

Taking the inverse z-transform, we obtain:

$$x[n] = \frac{2}{9}u[n] + \frac{7}{9}(-2)^n u[n]$$

Formula Sheet

S. No.	Continuous-Time	Discrete-Time
1.	$Frequency: \ f = \frac{1}{T}$ $Angular\ Frequency: \ \omega = 2\pi f = \frac{2\pi}{T}\ or\ f = \frac{\omega}{2\pi}$ $Fundamental\ Period: \ T = \frac{2\pi}{\omega}$	$Angular\ Frequency: \ \omega = \frac{2\pi k}{N}$ $Fundamental\ Period: \frac{N}{k} = \frac{2\pi}{\omega}$
2.	Convolution Integral $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$	Convolution Sum $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
	Fourier Series $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\ , Synthesis\ Equation$	Fourier Series $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}\ , Synthesis\ Equation$
3.	$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt\ , Analysis\ Equation$	$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n}\ , Analysis\ Equation$

Properties of the Discrete Time Fourier Transform

Non-periodic signal	*Fourier transform*
$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$	$X(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
$\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$	$\left.\begin{matrix} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{matrix}\right\}$ Periodic with period 2π
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
$x^*[n]$	$X^*(e^{j(-\omega)})$
$x[-n]$	$X(e^{j(-\omega)})$
$x_{(m)}[n] = \begin{cases} x[n/m], & n \text{ multiple of } m \\ 0, & n \text{ not multiple av } m \end{cases}$	$X(e^{j(m\omega)})$
$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
$x[n] - x[n-1]$	$\left(1 - e^{j\omega}\right) X(e^{j\omega})$
$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{j}\omega}X(e^{j\omega}) + \pi X(0)\sum_{k=-\infty}^{\infty}\delta(\omega - 2\pi k)$
$nx[n]$	$j\frac{d}{d\omega}X(e^{j\omega})$

If $x[n]$ is real valued then

$x[n]$	$\begin{cases} X(e^{j\omega}) = X^*(e^{j(-\omega)}) \\ \Re\{X(e^{j\omega})\} = \Re\{X(e^{j(-\omega)})\} \\ \Im\{X(e^{j\omega})\} = -\Im\{X(e^{j(-\omega)})\} \\ \left\|X(e^{j\omega})\right\| = \left\|X(e^{j(-\omega)})\right\| \\ \arg\{X(e^{j\omega})\} = -\arg\{X(e^{j(-\omega)})\} \end{cases}$
$x_e[n] = \mathcal{E}\{x[n]\}$	$\Re\{X(e^{j\omega})\}$
$x_o[n] = \mathcal{O}\{x[n]\}$	$j\Im\{X(e^{j\omega})\}$

Parsevals relation for non-periodic signals

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Basic Pairs of Discrete Time Fourier Transform

$x[n]$	$X(e^{j\omega})$
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\omega n_0}$
$\sum_{k=-\infty}^{\infty} \delta(n-kN)$	$\frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$
1	$2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$
$e^{j\omega_o n}$	$2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_o - 2\pi k)$
$\cos \omega_o n$	$\pi \sum_{k=-\infty}^{\infty} [\delta(\omega - \omega_o - 2\pi k) + \delta(\omega + \omega_o - 2\pi k)]$
$\sin \omega_o n$	$\frac{\pi}{j} \sum_{k=-\infty}^{\infty} [\delta(\omega - \omega_o - 2\pi k) - \delta(\omega + \omega_o - 2\pi k)]$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$
$a^n u(n), \quad \lvert a \rvert < 1$	$\frac{1}{1-ae^{-j\omega}}$
$(n+1)a^n u[n], \quad \lvert a \rvert < 1$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{(n+m-1)!}{n!(m-1)!} a^n u[n], \quad \lvert a \rvert < 1$	$\frac{1}{(1-ae^{-j\omega})^m}$
$\frac{1}{1-a^2} a^{\lvert n \rvert}, \quad \lvert a \rvert < 1$	$\frac{1}{1+a^2-2a\cos\omega}$
$\begin{cases} 1, & \lvert n \rvert \le N_1 \\ 0, & N_1 < \lvert n \rvert \le \frac{N}{2} \end{cases}$ period N	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega \frac{2\pi k}{N}\right)$
$\begin{cases} 1, & \lvert n \rvert \le N_1 \\ 0, & \lvert n \rvert > N_1 \end{cases}$	$\frac{\sin \omega \left(N_1 + \frac{1}{2}\right)}{\sin \frac{\omega}{2}}$
$\begin{cases} \frac{\sin Wn}{\pi n} = \frac{W}{\pi} \text{sinc} \frac{Wn}{\pi} \\ 0 < W < \pi \end{cases}$	$\begin{cases} 1, & \lvert \omega \rvert \le W \\ 0, & W < \lvert \omega \rvert \le \pi \end{cases}$ period 2π

Z-Transform Pairs

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9. $[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

Z-Transform Properties

signal	*Z-transform*	*ROC*
$x[n]$	$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n]z^{-n}$	R_x
$ax[n] + by[n]$	$aX(z) + bY(z)$	Contains $R_x \cap R_y$
$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except possible addition or deletion of the origin or ∞
$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
$x^*[n]$	$X^*(z^*)$	R_x
$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
$x[n] * y[n]$	$X(z)Y(z)$	Contains $R_x \cap R_y$
$nx[n]$	$-z\frac{d}{dz}X(z)$	R_x, except possible addition or deletion of the origin or ∞
$\Re\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
$\Im\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x

Initial value theorem

$x[n] = 0,\ n < 0 \quad \lim_{z\to\infty} X(z) = x[0]$