Calculus-II

Lecture #2

Separable Differential Equations

26th Feb, 2019

Concept of Solution

Solution of ODE

- A function $y=h(x)$ is called a solution of ODE $y'=f(x,y)$, on some open interval $a < x < b$ if $h(x)$ is defined and differentiable throughout the interval.
- The curve (graph) of h is called a solution curve.
- Here, open interval $a < x < b$ means that the endpoints a and b are not regarded as points belonging to the interval.
- Also $a < x < b$ includes infinite intervals $-\infty < x < b$, $a < x < \infty$, $-\infty < x < \infty$ as special cases.

Example #1

- Verify that $y = c/x$ (c an arbitrary constant) is a solution of the ODE $xy' = -y$.

Solution of ODE (cont.)

- The ODE in the above example has a solution that contains an arbitrary constant c. Such a solution is known as general solution of the ODE.
- The solution that does not contain any arbitrary constant is known as particular solution of the ODE.

Initial Value Problem

- The unique solution i.e., a Particular solution is obtained from a general solution by an initial condition $y(x_0) = y_0$, with given values of x_0 and y_0 , that is used to determine a value of the arbitrary constant c.
- Geometrically this condition means that the solution curve should pass through the point (x_0 , y_0) in the xy-plane.
- An ODE with an initial condition is known as initial value problem.
- Thus, if the ODE is explicit, y' = f(x,y), the initial value problem is of the form:

$$y' = f(x, y), \quad y(x_0) = y_0$$

Example #2

- Solve the initial value problem:

$$y' = \frac{dy}{dx} = 3y, \quad y = ce^{3x}, \quad y(0) = 5.7$$

Example #3

- Solve the initial value problem:

$$y' = 0.5y, \quad y = ce^{0.5x}, \quad y(2) = 2$$

Singular Solution

- An ODE may sometimes have an additional solution that cannot be obtained from the general solution is then called a singular solution.

Example #4

- The ODE $(y')^2 - xy' + y = 0$ is of the kind.
- Show by differentiation and substitution that it has the general solution $y = cx - c^2$ and the singular solution $y = \dfrac{x^2}{4}$.

Separable ODEs

Separable ODE

- An ODE is said to be separable if the variables can be separated e.g.,

$$g(y)dy = f(x)dx$$

- Let's say we have a first order differential equation reduced to the form:

$$g(y)y' = f(x) \rightarrow (1)$$

- Then we can integrate on both sides with respect to x, obtaining:

$$\int g(y)y'\,dx = \int f(x)\,dx + c \rightarrow (2)$$

Separable ODE (cont.)

- On the left we can switch to y as the variable of integration. By calculus, y' dx = dy, so that:

$$\int g(y)\,dy = \int f(x)\,dx + c \rightarrow (3)$$

- If f and g are continuous functions, the integrals for above equation exist and by evaluating them we obtain the general solution of (equ. 1).

- This method of solving ODEs is called the method of separating variables and (equ. 1) is called a separable equation.

- In (equ. 3) the variables are now separated: x appears only on the right and y only on the left.

Example #5

- Solve the equation:

$$2ydy = \left(x^2 + 1\right)dx$$

Reduction to Separable Form

Reduction to Separable Form

- There are certain ODE which seems non-separable but we can make them separable by transformation i.e., by introducing a new unknown function.
- Let's say we have an ODE: y' = f(y/x)
- Here , the function f is any differentiable function of (y/x) such as $(y/x)^4$, sin (y/x) and so on.
- Now in order to solve such an ODE we set y/x = u; thus,

$$y = ux \quad and \quad by \quad product \quad differentiation \quad y' = u'x + u$$

Reduction to Separable Form (cont.)

- Now substitute y’ and u in equation y’ = f(y/x):

$$u'x + u = f(u)$$

$$u'x = f(u) - u$$

- Here f(u) - u ≠ 0, this can be separated.

$$x\frac{du}{dx} = f(u) - u$$

$$xdu = \left[f(u) - u\right]dx$$

$$\frac{du}{f(u) - u} = \frac{1}{x}dx$$

Reduction to Separable Form (cont.)

- Hence both the variable are now separated.

Example #6

- Solve:

$$\frac{dy}{dx}+\frac{1}{2}y=\frac{3}{2}, \quad y(0)=2$$

Example #7

- Solve:

$$2xyy' = y^2 - x^2$$

Exercise Problems

Problem #1

- Find a general solution. Show the steps of derivation.

$$y' + \left(x + 2\right) y^2 = 0$$

Problem #2

- Solve the IVP. Show steps of derivation, beginning with the general solution:

$$yy' + 4x = 0, \quad y(0) = 3$$

The End