Calculus-II

Lecture #3

Exact Differential Equations

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Exact Differential Equations

Exact ODEs

- An Exact differential equation is also known as total differential equation.
- If a function u(x,y) has continuous partial derivatives its differential is:

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

- If u(x,y)=c=constant, then du=0.
- A first order ODE M(x,y)+N(x,y)y’=0, and dy=y’dx then:

Exact ODEs (cont.)

$$M\left(x,y\right)dx+N\left(x,y\right)dy=0\rightarrow\left(1\right)$$

- Is known as exact differential equation if the differential equation M(x,y)dx + N(x,y) dy is exact.
- This form is the differential $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy\rightarrow\left(2\right)$ of some function u(x,y) then du=0.
- By integrating we obtain the general solution of equ. (1) i.e., u(x,y) = c. This is known as implicit solution.
- Comparing equ. (1) and equ. (2) we see that (1) is an exact differential equation if there is some function u(x,y), such that:

$$\left(a\right):\quad\frac{\partial u}{\partial x}=M,\quad\left(b\right):\quad\frac{\partial u}{\partial y}=N\quad\rightarrow\left(4\right)$$

Theorem

- If $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous, then the differential equation:

$$M(x,y)dx + N(x,y)dy = 0$$

- Is exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
- **Proof:**
- To prove the theorem, let us assume first that the given equation is exact. Under this assumption there exists a function u(x,y) such that:

Theorem (cont.)

$$M = \frac{\partial u}{\partial x} \quad and \quad N = \frac{\partial u}{\partial y}$$

- Hence,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad and \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

- By the assumption of continuity the two second partial derivatives are equal. Thus:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow (5)$$

- This equation not only necessary but also sufficient for equ.(1) to be an exact differential equation.

Theorem (cont.)

- If equ. (1) is exact, the function can be found by inspection or in the following systematic way.
- Let us first integrate M(x,y) with respect to x, holding y fixed then:

$$u(x,y) = \int M\,dx + k(y)$$

- k(y) is the constant of integration and y is regarded as constant.
- To determine k(y) we derive $\frac{\partial u}{\partial y}$ from above equation we use $\frac{\partial u}{\partial y} = N$ to get dk/dy.

Theorem (cont.)

- Then we integrate dk/dy to get k.

Example #1

- Solve:

$$\cos(x+y)\,dx+\left(3y^2+2y+\cos(x+y)\right)dy=0$$

Example #2

- Solve:

$$\left(3xy - y^2\right)dx + x\left(x - y\right)dy = 0$$

Example #3

- Solve Initial Value Problem (IVP):

$$\left(3x^2y-1\right)dx+\left(x^3+6y-y^2\right)dy=0,\quad y\left(0\right)=3$$

Integrating Factors

Reduction to Exact Form

- If the differential equation $M(x,y)dx + N(x,y)dy = 0$ is not exact then we multiply it by a factor which may be a function of x alone or a function of y alone or a function of both x & y and then the resulting differential equation become exact.
- That multiplying factor is known as Integrating factor (I.F).

How to find an Integrating Factor?

- Assume that the equation $M(x,y)dx + N(x,y)dy = 0 \rightarrow (1)$ is not exact, i.e., $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.
- Then in this case we multiply equ. (1) with I.F. Let suppose in this case we have q(x,y) as an integrating factor.
- That is: $q(x,y)\big(M(x,y)dx + N(x,y)dy\big) = 0$
- Now this equation is Exact, then the function q(x,y) is called the integrating factor.
- Now q(x,y) satisfies the following condition:

$$\frac{\partial M}{\partial y}q + \frac{\partial q}{\partial y}M = \frac{\partial N}{\partial x}q + \frac{\partial q}{\partial x}N \rightarrow (2)$$

How to find an Integrating Factor? (cont.)

- Equ. (2) is not an ordinary differential equation as it involves more than one variable, hence known as partial differential equation.
- In this case we have 2 special cases.
- **Case #1:**
  - There exists an I.F q(x) function of x only. This happens if the expression:
    $$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$$
  - Above equation is a function of x only, hence the variable y disappears from the expression.

How to find an Integrating Factor? (cont.)

- In this case the function q is given by:

$$q(x) = \exp\left[\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx\right]$$

- **Case #2:**
  - There exists an I.F q(y) function of y only. That happens if the expression: $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$
  - Is a function of y only, that is the variable x disappears from the expression.
  - In this case the function q is given by: $q(y) = \exp\left[\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dy\right]$

How to find an Integrating Factor? (cont.)

- Once the factor is found, multiply it with the old equation to get a new one which is Exact.
- Then use the technique of Exact differential equation to solve it.

Example #4

- Solve:

$$\left(e^{x+y} + ye^{y}\right)dx + \left(xe^{y} - 1\right)dy = 0, \quad y\left(0\right) = -1$$

Example #5

- Solve:

$$-ydx + xdy = 0$$

The End