BSC EngTel

day / date: Tue/12 March, 19

LECTURE #4

-: SOLVED EXAMPLES :-

EXAMPLE #1:-

$$y' - y = e^{2x}$$

Sol:-

$$y' + p(x)y = r(x)$$

$$p = -1 \quad , \quad r = e^{2x}$$

$$h = \int p\,dx = \int -1\,dx \Rightarrow -x$$

Using $y = e^{-h}\left(\int e^{h} r\,dx + c\right)$

$$y(x) = e^{+x}\left[\int e^{-x} e^{2x}\,dx + c\right]$$

$$= e^{x}\left[\int e^{-x+2x}\,dx + c\right]$$

$$= e^{x}\left[\int e^{x}\,dx + c\right]$$

$$y(x) = e^{x}\left[e^{x} + c\right]$$

$y(x) = e^{2x} + ce^{x}$ → general solution.

EXAMPLE #2:-

$$y' + y\tan x = \sin 2x \quad , \quad y(0) = 1$$

Sol:-

$$y' + p(x)y = r(x)$$

$$p = \tan x \quad , \quad r(x) = \sin 2x$$

$$h = \int p\,dx = \int \tan x\,dx$$

$$h = \ln|\sec x|$$

Now using formula $y(x) = e^{-h}\left[\int e^{h} r \, dx + c\right]$

as $h = \ln|\sec x|$

~~$e^{h} = \ln|\sec x|$~~

$e^{-h} = e^{-\ln|\sec x|}$ $\quad\therefore$ exponent rule $a^{bc} = (a^b)^c$

$e^{-h} = e^{(\ln|\sec x|)(-1)}$

Apply log rule $a^{\log_a(b)} = b$

$= e^{\ln(\sec(x))} = \sec(x)$

$= \sec^{-1}(x) \Rightarrow \dfrac{1}{\sec x}$ $\quad\therefore \dfrac{1}{\sec x} = \cos x$

$\Rightarrow e^{-h} = \cos x$

$\Rightarrow e^{h} = e^{\ln|\sec x|} \Rightarrow \sec x$

And as $\sin 2x = 2\sin x \cos x$ then,

$r = 2\sin x \cos x$

$y(x) = \cos x\left[\int \sec x\,(2\sin x \cos x)\,dx + c\right]$

$= \cos x\left[2\int \dfrac{1}{\cancel{\cos x}}(\sin x \cancel{\cos x})\,dx + c\right]$

$= \cos x\left[2\int \sin x \, dx + c\right]$

$= \cos x\left[2(-\cos x) + c\right]$

$y(x) = c\cos x - 2\cos^2 x$

Now $y(0) = 1$

$$y(0) = c\cos(0) - 2\cos^2(0) = 1$$

$$c - 2(1) = 1$$
$$c = 3$$

then $y = 3\cos x - 2\cos^2 x$ ⇒ particular solution.

Example #3:-

$$y' = Ay - By^2 \rightarrow (a)$$

Sol:-

$$y' - Ay = -By^2$$

Now as $a = 2$, so that $v = y^{1-a} \Rightarrow y^{-1}$

Now differentiate and ~~substitute~~ multipl y' from (a)

$$v = y^{-1}$$
$$v' = -y^{-1-1} \Rightarrow -y^{-2}$$

Now,

$$v' = -y^{-2}(y') = -y^{-2}(Ay - By^2)$$

$$= -y^{-2}(y') = -Ay^{1-2} + By^{-2+2}$$
$$= -y^{-2}(y') = -Ay^{-1} + By^{0}$$

$$= -y^{2}(y') = -Ay^{-1} + B$$

let $y' = v$ then

$v' = B - Av$ is the linear ODE

day / date:

then , $B = v' + Av$

$$v = e^{-h}\int e^{h} r\,dx + ce^{-h}$$

$$r = B \quad , \quad p(t) = A$$

$$h = \int p(t)\,dt \Rightarrow At$$

$$v = e^{-At}\left[\int e^{At} B\,dt + C\right]$$

$$v = e^{-At}\left[\frac{e^{At}}{A} B + C\right]$$

$$v = \frac{e^{-At+At}}{A} B + e^{-At} C$$

$$v = \frac{B}{A} + e^{-At} C \quad \text{general solution}$$

Now as $v = \frac{1}{y}$ then

$$y = \frac{1}{v} = \frac{1}{Ce^{-At} + B/A} \quad \text{solution}$$

EXAMPLE # 4:-

$$y' + ty = ty^3 \rightarrow (1)$$

Sol:-

linear equation :- $y' + p(x)y = r(x)$

Now as we see $a = 3$ then,

$v = y^{1-a} \Rightarrow y^{1-3} \Rightarrow y^{-2}$

Now differentiate it and multiply with equ (1) ~~$y' + p(x)y = y(t)y$~~

$v' = -2y^{-2-1} \Rightarrow -2y^{-3}$

$y' + ty = ty^3$

$v' = (-2y^{-3})y' + (-2y^{-3})ty = (-2y^{-3})ty^3$

$v' = (-2y^{-3})y' - 2y^{-3+1}t = -2y^{-3+3}t$

$v' = (-2y^{-3})y' - 2y^{-2}t = -2y^{0}t$

$= (-2y^{-3})y' = 2y^{-2}t - 2t \rightarrow$

$v' = 2y^{-2}t - 2t \rightarrow (2)$

As $v = y^{-2}$ then eqn (2) becomes

$v' = 2vt - 2t$

$v' - 2vt = -2t$ linear ODE transformed

Now, $r = -2t$, $p(t) = -2t$ then

$h = e^{\int p(t)dt} = e^{\int -2t\,dt} = e^{-2\frac{t^2}{2}} \Rightarrow e^{-t^2}$ then

$y = e^{-h}\left[\int e^{h} r\,dt + c\right]$

$\cancel{= e^{t^2}} = e^{-(-t^2)}\left[\int e^{-t^2}(-2t)\,dt + c\right]$

$= e^{t^2}\left[\cancel{-2t}\int e^{-t^2}dt + c\right]$

$= e^{t^2}\left[-2t\int e^{-t^2}dt + c\right] \rightarrow (2)$

$\therefore \int e^{-t^2}dt$ let $t^2 = u$

$du = 2t\,dt$

$dt = \frac{du}{2t}$

$$\int e^{-u} \frac{du}{2t} \Rightarrow -\frac{e^{-u}}{2t}$$

$\frac{-e^{-t^2}}{2t}$ put in equ ②

$$= e^{t^2}\left[2t\left[-\frac{e^{-t^2}}{2t}\right]+c\right]$$

$$y = -e^{t^2-t^2} + ce^{t^2}$$

$y = -1 + ce^{t^2}$ general solution.

day / date:

PROBLEM # 1:

$$y' + 1.25y = 5 \ , \quad y(0) = 6.6$$

Sol:

As R.H.S is not zero that means it is non-homogenous linear ODE then use formula:-

$$y(x) = e^{-h}\left[\int e^{h} r \, dx + c\right]$$

as $y' + p(x)y = r(x)$

$$y' + 1.25y = 5$$

$p = 1.25 \quad , r(x) = 5$

$h = \int p\,dx \Rightarrow \int 1.25\,dx \Rightarrow 1.25x$

$$y(x) = e^{-1.25x}\left[\int e^{1.25x}(5)\,dx + c\right]$$

$$= e^{-1.25x}\left[5\int e^{1.25x}\,dx + c\right]$$

$$= e^{-1.25x}\left[5e^{1.25x}\left(1/1.25\right) + c\right]$$

$$y(x) = e^{-1.25x}\left[4\,e^{1.25x} + c\right]$$

$$= 4\,e^{\cancel{-1.25x+1.25x}} + c\,e^{-1.25x}$$

$y(x) = 4 + ce^{-1.25x}$ general solution

Now $y(0) = 6.6$

$$y(0) = 4 + ce^{-1.25(0)}$$

$$6.6 = 4 + c$$

$$c = 2.6$$

$$y = 4 + 2.6e^{-1.25x} \quad \text{particular solution.}$$

PROBLEM #2:-

$$y' = 5.7y - 6.5y^2$$

Sol:

$$y' = 5.7y - 6.5y^2$$

$$y' - 5.7y = -6.5y^2 \Rightarrow (1)$$

Now $a = 2$ then $v = y^{1-a} = y^{1-2} \Rightarrow y^{-1}$
Now differentiate and put in equ (1)

$$v' = -1(y^{-2})$$

$$v' - (-y^{-2})5.7y = (-y^{-2})(-6.5y^2)$$

$$v' + y^{-2+1}5.7 = -y^{-2+2}6.5$$

$$v' + y^{-1}5.7 = 6.5 \quad \because v = y^{-1}$$

$$v' + v5.7 = 6.5$$

$$p = 5.7, \quad r = 6.5$$

$$h = \int p\,dx \Rightarrow 5.7x$$

$$v = e^{-5.7x}\left[\int e^{5.7x}(6.5)\,dx + c\right]$$

$$= e^{-5.7x}\left[\frac{e^{5.7x}}{5.7}(6.5) + c\right]$$

$$v = e^{-5.7x}\left[1.140e^{5.7x} + C\right]$$

$$v = e^{\cancel{-5.7x+5.7x}}\, 1.140 + Ce^{-5.7x}$$

$$v = 1.140 + Ce^{-5.7x}$$

Now $v = y^{-1}$, $y = v^{-1}$

$$y = \frac{1}{1.140 + Ce^{-5.7x}}$$ general solution.