Calculus-II

Lecture #4

Linear ODEs & Bernoulli's Equation

12th Mar, 2019

Linear ODEs

Linear ODEs

- A first order ODE is said to be linear if it can be written into the form: $$y' + p(x)y = r(x)$$
- This equation is linear in both unknown function y and its derivative i.e., y' = dy/dx. And p and r may be any given function of x.
- A first order ODE is said to be non-linear if it cannot be written in the above form.
- If the first term is f(x)y' instead of y, divide the equation by f(x) to get the standard form with y' as the first term.

Linear ODEs (cont.)

- For example: $y' \cos x + y \sin x = x$ is linear ODE but for its standard form we will divide cos x with the equation then:

$$y' \frac{\cos x}{\cos x} + y \frac{\sin x}{\cos x} = x \frac{1}{\cos x} \Rightarrow y' + y \tan x = x \sec x$$

- Is now the standard form.
- **Homogeneous Linear ODE:**
  - Lets suppose we want to solve y' + p(x) y = r(x) in some interval a< x <b, call it J and we begin with the simple case that r(x) is zero for all x in J.

Linear ODEs (cont.)

- **Homogeneous Linear ODE:**
  - And if we say that c=0 then we obtain trivial solution i-e., y(x)=0 for all x in that interval.
- **Non-Homogeneous Linear ODE:**
  - Now if we consider the case in which $r(x)\neq 0$ in the interval J, then the ODE is known as non-homogeneous.
  - So in that case linear equation has an integrating factor depending only on x.
  - We can find the integrating factor by case 1 as we discussed in previous lecture.

Linear ODEs (cont.)

- **Non-Homogeneous Linear ODE:**
  - Separation and integration gives:

$$\frac{dF}{F} = p\,dx \Rightarrow \ln|F| = \int p\,dx$$

  - Taking exponents on both sides we get out I.F:

$$F(x) = e^{\int p\,dx}$$

$$e^{\int p\,dx}\left[y' + py\right] = e^{\int p\,dx} r$$

$$\left(e^{\int p\,dx} y\right)' = e^{\int p\,dx} r$$

Linear ODEs (cont.)

- **Non-Homogeneous Linear ODE:**
  - Now integrate w.r.t x:

$$e^{\int p\,dx} y = \int e^{\int p\,dx} r\,dx + c$$

  - Divide equ. by $e^{\int p\,dx}$ we get:

$$y = e^{-h}\left(\int e^{h} r\,dx + c\right) \quad \therefore h = \int p\,dx$$

$$y(x) = e^{-h}\int e^{h} r\,dx + ce^{-h}$$

Example #1

- Solve the following ODE:

$$y' - y = e^{2x}$$

Example #2

- Solve the following ODE:

$$y' + y\tan x = \sin 2x, \quad y(0) = 1$$

Bernoulli Equation

Reduction to Linear Form

- To transform a nonlinear ODE into linear ODE we use Bernoulli equation.
- A differential equation of the form $y' + p(x)y = g(x)y^a$ is called the Bernoulli's equation. Where a can be any real number.
- If a=0 or a=1, then the above equation is linear. Otherwise it is nonlinear.
- Then we set:

$$u(x) = \left[y(x)\right]^{1-a}$$

Reduction to Linear Form (cont.)

- We differentiate above equation and substitute y' from equ(1) gives:

$$u' = (1-a)\,y^{-a}\,y' = (1-a)\,y^{-a}\left(gy^{a} - py\right)$$

$$u' = (1-a)\,y^{-a} \quad multiply \quad it \quad with \quad main \quad equation$$

$$u' = (1-a)\,y^{-a}\,y' + (1-a)\,y^{-a}\,p(x)\,y = (1-a)\,y^{-a}\,g(x)\,y^{a}$$

$$u' = (1-a)\,y^{-a}\,y' = (1-a)\,y^{-a}\left[gy^{a} - py\right]$$

- Simplification gives:

$$u' = (1-a)\left[g - py^{1-a}\right]$$

Reduction to Linear Form (cont.)

- Where $y^{1-a} = u$ on the right so we get the linear ODE:

$$u' = (1-a)\left[g - pu\right]$$

$$u' + (1-a)\,pu = (1-a)\,g$$

Example #3

- Solve the Bernoulli's equation known as the logistic equation:

$$y' = Ay - By^2$$

Problems

Problem #1

- Solve the given ODE:

$$y' + 1.25\,y = 5, \quad y\left[0\right] = 6.6$$

Problem #2

- Solve the given ODE:

$$y' = 5.7\,y - 6.5\,y^2$$

The End