Calculus-II

Lecture #5

Homogeneous Linear Differential Equation

26^{th} Mar, 2019

Linear ODE of First Order

Example #1

- Solve the Linear ODE:

$$y' + 1.25y = 5, \quad y(0) = 6.6$$

Bernoulli Equation

Reduction to Linear Form

- To transform a nonlinear ODE into linear ODE we use Bernoulli equation.
- A differential equation of the form $y' + p(x)y = g(x)y^a$ is called the Bernoulli's equation. Where a can be any real number.
- If a=0 or a=1, then the above equation is linear. Otherwise it is nonlinear.
- Then we set:

$$u(x) = [y(x)]^{1-a}$$

Reduction to Linear Form (cont.)

- We differentiate above equation and substitute y' from equ(1) gives:

$$u' = \left(1-a\right) y^{-a} y' = \left(1-a\right) y^{-a} \left(gy^{a} - py\right)$$

$$u' = \left(1-a\right) y^{-a} \quad multiply \quad it \quad with \quad main \quad equation$$

$$u' = \left(1-a\right) y^{-a} y' + \left(1-a\right) y^{-a} p\left(x\right) y = \left(1-a\right) y^{-a} g\left(x\right) y^{a}$$

$$u' = \left(1-a\right) y^{-a} y' = \left(1-a\right) y^{-a} \left[gy^{a} - py\right]$$

- Simplification gives:

$$u' = \left(1-a\right)\left[g - py^{1-a}\right]$$

Reduction to Linear Form (cont.)

- Where $y^{1-a} = u$ on the right so we get the linear ODE:

$$u' = (1-a)\left[g - pu\right]$$

$$u' + (1-a)\,pu = (1-a)\,g$$

Example #2

- Solve the Bernoulli's equation:

$$y' + ty = ty^3$$

Example #3

- Solve the Bernoulli's equation:

$$y' + x^2 y = \frac{\left(e^{-x^3}\sinh x\right)}{3y^2}$$

Homogeneous Linear ODEs of Second Order

Linear ODEs of Second Order

- A second-order ODE is called linear if it can be written:

$$y'' + p(x)\,y' + q(x)\,y = r(x)$$

- And non-linear if it cannot be written in this form.
- The above equation is linear in y and its derivatives, whereas the function p, q, and r on the right may be any given functions of x.
- If the equation begins with say f(x)y'', then divide by f(x) to have the standard form with y'' as the first term.

Linear ODEs of Second Order (cont.)

- If in the above equation $r(x) = 0$ then it is called homogeneous.
- If $r(x) \neq 0$ then the equation is called nonhomogeneous.

Superposition Principle

- For the homogeneous equation the backbone of this structure is the superposition principle or linearity principle.
- It states that we can obtain further solution from given ones by adding them or by multiplying them with any constants.

Example #4

- The functions y=cos(x) and y=sin(x) are the solutions of the homogeneous linear ODE y'' + y=0 for all x. Verify this by differentiation and substitution.

Fundamental Theorem for the Homogeneous Linear ODE

- For a homogeneous linear ODE i.e., $y'' + p(x)y' + q(x)y = 0$, any linear combination of two solutions on an open interval I is again a solution of $y'' + p(x)y' + q(x)y = 0$ on I. in particular for such an equation sums and constant multiples of solution are again solution.
- Note: This theorem hold for homogeneous linear ODEs only but does not hold for non-homogeneous linear or nonlinear ODEs.

Initial Value Problem

- We are given the initial condition which is used to determine the arbitrary constant c in the general solution of the ODE. This result is a unique solution and known as a particular solution of the ODE.
- But in this case we will consider for second order homogeneous linear ODE an initial value problem consists of two initial conditions.
- That is $y_0(x_0) = k_0$, $y'(x_0) = k_1$ these conditions are used to determine arbitrary constant c_1 and c_2 in a general solution: $y=c_1y_1 + c_2y_2$.

Example #5

- Let's consider an example:

$$y'' + y = 0, \quad y(0) = 6.6, \quad y'(0) = -0.5$$

The End