Example #1

Solve

$M \qquad \checkmark \qquad N$

$$\cos(x+y)\,dx + \left(3y^2+2y+\cos(x+y)\right)dy = 0 \rightarrow (1)$$

Solution:-

Step 1:- Test for exactness. so,

$$M = \cos(x+y) \quad , \; N = 3y^2+2y+\cos(x+y)$$

Thus

$$\frac{\partial M}{\partial y} = -\sin(x+y)$$

$$\frac{\partial N}{\partial x} = -\sin(x+y)$$

From this it's clear that (1) is exact.

Step 2:- Implicit general solution.

Now as $U(x,y) = c = \int M$

$$U = \int M\,dx + k(y)$$

$$= \int \cos(x+y)\,dx + ky \Rightarrow \sin(x+y) + k(y) \rightarrow (2)$$

To find $k(y)$ we will differentiate this formula with respect to y and use formula $\frac{\partial U}{\partial y} = N$

$$\frac{\partial u}{\partial y} = \cos\cancel{(x+y)} + \frac{dk}{dy} = N = 3y^2 + 2y + \cos\cancel{(x+y)}$$

Hence $\dfrac{dk}{dy} = 3y^2 + 2y$

Now by integration $k = y^3 + y^2 + C_1$
insert this in equation ② and observing & $u(x,y) = c$
we get

$$u(x,y) = \sin(x+y) + y^3 + y^2 \pm C. \quad \text{ans} \qquad (C = -C_1)$$

Step 3:- Checking an implicit solution.

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$= \cos(x+y)\,dx + \{\cos(x+y) + 3y^2 + 2y)\,dy = 0$$

This completes the check.

⇒ and if we are given initial condition as $y(1) = 2$ then put it in general solution

$$u(x,y) = \sin(x+y) + y^3 + y^2 = C$$

$$= \sin(1+2) + (2)^3 + (2)^2 = C$$

$$\sin(3) + 8 + 4 = C$$

$$C = 12.05.$$

so $\sin(x+y) + y^3 + y^2 = 12.05$ is particular solution.

Example #2

$$(3xy - y^2)\,dx + x(x-y)\,dy = 0 \rightarrow ①$$

$$M = 3xy - y^2, \quad N = x^2 - xy$$

$$\frac{\partial M}{\partial y} = 3x - 2y, \quad \frac{\partial N}{\partial x} = 2x - y.$$

As $M \neq N$ then its clear that equ① is not an exact differential equation.

Example #3

Solve IVP :-

~~If~~ $(3x^2y-1)dx + (x^3+6y-y^2)dy = 0$, $y(0)=3$

$M = 3x^2y - 1$, $N = x^3+6y-y^2$

$\frac{\partial M}{\partial y} = 3x^2$, $\frac{\partial N}{\partial x} = 3x^2$

Hence both ~~M ≠~~ M = N then the equ ① is exact differential equation.
So let's find its general solution.

$U = \int M dx + k(y)$

$U = \int (3x^2y-1)\,dx + k(y) \Rightarrow x^3y - x + k(y) \rightarrow ②$

Now to find $k(y)$ we will differentiate equ ② w.r.t y.

$\frac{\partial U}{\partial y} = x^3 + k'$

Now as $N = \frac{\partial U}{\partial y}$ then

$$x^3 + 6y - y^2 = x^3 + k'$$

$$k' = \frac{dk}{dy} = 6y - y^2$$

~~$u(x,y) = x^3y - x$~~ Now integrate k'

$$k(y) = 3y^2 - \frac{1}{3}y^3 + c$$

So, general solution will be. $\therefore -c = C$

$$u(x,y) = x^3y - x + 3y^2 - \frac{1}{3}y^3 = C.$$

Now find out its particular solution by $y = 3,\ x = 0$

$$C = (0)^3(3) - (0) + 3(3)^2 - \frac{1}{3}(3)^3$$

$$C = 27 - 9 \Rightarrow 18$$

Then particular solution will be:-

$$x^3y - x + 3y^2 - \frac{1}{3}y^3 = 18. \quad \text{ans}$$

Example #4

$$(e^{x+y} + ye^y)\,dx + (xe^y - 1)\,dy = 0 \rightarrow (1), \quad y(0) = -1$$

Solution:- Step: Check for exactness :-

$$M(x,y) = e^{x+y} + ye^y, \quad N(x,y) = xe^y - 1$$

$$\frac{\partial M}{\partial y} = e^{x+y} + ye^y + e^y, \quad \frac{\partial N}{\partial x} = e^y$$

as $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then the given D.E. is not exact.

then use the case 1 or case 2 to find out the I.F.

Case 1:- $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = e^{x+y} + ye^y + e^y - \cancel{e^y}$

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{xe^y - 1}\left(e^{x+y} + ye^y\right)$$

As the R.H.S equation depends on both x and y then try case 2:-

Case 2:- $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{1}{e^{x+y} + \cancel{ye^y}}\left(\cancel{e^y} - \cancel{e^{x+y}} - \cancel{ye^y} - \cancel{e^y}\right)$

$$q(x,y) = -1$$

Now integrate it and take exponent.

$q(y) = e^{\int -1\, dy} \Rightarrow e^{-y}$ Now multiply it with equ(1)

i-e,

$$e^{-y}\left(e^{x+y} + ye^y\right)dx + e^{-y}\left(xe^{+y} - 1\right)dy = 0$$

$$\left(e^{x+y-y} + ye^{y-y}\right)dx + \left(xe^{-y+y} - e^{-y}\right)dy = 0$$

$$\left(e^x + y\right)dx + \left(x - e^{-y}\right)dy = 0 \rightarrow (2)$$

Now check again for exactness :-

$M(x,y) = e^{x} + y \quad , \quad N(x,y) = x - e^{-y}$

$\frac{\partial M}{\partial y} = 1 \quad , \quad \frac{\partial N}{\partial x} = 1$

Hence both $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then solve it with the method of exact D.E.

$$U(x,y) = \int M(x)\, dx + k(y)$$

$$= \int (e^x + y)\, dx + k(y) \Rightarrow e^x + yx + k(y) \rightarrow (3)$$

Now differentiate w.r.t y and compare with N.

$$\frac{\partial U}{\partial y} = 0 + x + k'(y) = N = x - e^{-y}$$

$$k'(y) = -e^{-y}$$

Date:

Now integrate it,

$$K(y) = -\int e^{-y}\,dy \Rightarrow e^{-y} + C^{*}$$ put it in eq ③

$$U(x,y) = e^{x} + yx + e^{-y} + C^{*} = C_1$$

$$e^{x} + yx + e^{-y} = C$$

Now ~~the~~ for particular solution.

$$y(0) = -1$$

$$e^{0} - 1(0) + e^{-(-1)} = C$$

$$C = e + 1 \Rightarrow 3.72 \text{ Then}$$

$$e^{x} + yx + e^{-y} = 3.72$$ is particular Solution.

Example #5

11-(2) :- $-y\,dx + x\,dy = 0 \rightarrow ①$

$$M(x,y) = -y \quad , \quad N(x,y) = x$$

Step1 :- $\dfrac{\partial M}{\partial y} = -1 \quad , \quad \dfrac{\partial N}{\partial x} = 1$

As $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ then ~~that~~ equ ① is not exact.

Then let's find out its I.F.

Try Case 1 :- $\dfrac{1}{N}\left[\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right] = \dfrac{1}{x}\left[-1-1\right] \Rightarrow \dfrac{-2}{x}$

then $q(x) = \exp\left(\int \dfrac{-2}{x}\,dx\right) \Rightarrow \exp^{-2\ln(x)}$

$q(x) = \exp^{\ln(x)^{-2}} \Rightarrow \dfrac{1}{x^2}$ multiply it with equ ①

$$\frac{1}{x^2}(-y)\,dx + \frac{1}{x^2}(x\,dy) = 0$$

$$-\frac{y}{x^2}\,dx + \frac{1}{x}\,dy = 0 \rightarrow ②$$

Now check for exactness.

$$M(x,y) = \frac{-y}{x^2} \quad , \quad N(x,y) = \frac{1}{x}$$

$$\frac{\partial M}{\partial y} = -\frac{1}{x^2} \quad , \quad \frac{\partial N}{\partial x} = x^{-1-1} \Rightarrow -\frac{1}{x^2}$$

Hence equ ② is exact as $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ then Solve

$$U(x,y) = \int N(y)\, dy + k(x)$$

$$= \int \frac{1}{x}\, dy + k(x) \Rightarrow \frac{y}{x} + k(x) \rightarrow (3)$$

Now differentiate w.r.t x and compare with M

$$-\frac{y}{x^2} + k'(x) = M^{\partial} = -\frac{y}{x^2}$$

$k'(x) = 0$ Now integrate it

$k(x) = C^*$ put in eqn (3)

$$U(x,y) = \frac{y}{x} + C^* = C_1$$

$\frac{y}{x} = C$ ans or $y = Cx$ ans