Calculus-II

Lecture #6

Homogeneous Linear Differential Equation

30th April, 2019

Homogeneous Linear ODEs of Second Order

Linear ODEs of Second Order

- A second-order ODE is called linear if it can be written:

$$y'' + p(x)y' + q(x)y = r(x)$$

- And non-linear if it cannot be written in this form.
- The above equation is linear in y and its derivatives, whereas the function p, q, and r on the right may be any given functions of x.
- If the equation begins with say f(x)y'', then divide by f(x) to have the standard form with y'' as the first term.

Linear ODEs of Second Order (cont.)

- If in the above equation $r(x) = 0$ then it is called homogeneous.
- If $r(x) \neq 0$ then the equation is called nonhomogeneous.

Superposition Principle

- For the homogeneous equation the backbone of this structure is the superposition principle or linearity principle.
- It states that we can obtain further solution from given ones by adding them or by multiplying them with any constants.

Example #1

- The functions y=cos(x) and y=sin(x) are the solutions of the homogeneous linear ODE y'' + y=0 for all x. Verify this by differentiation and substitution.

Fundamental Theorem for the Homogeneous Linear ODE

- For a homogeneous linear ODE i.e., $y'' + p(x)y' + q(x)y = 0$, any linear combination of two solutions on an open interval I is again a solution of $y'' + p(x)y' + q(x)y = 0$ on I. in particular for such an equation sums and constant multiples of solution are again solution.
- Note: This theorem hold for homogeneous linear ODEs only but does not hold for non-homogeneous linear or nonlinear ODEs.

Example #2

- Verify by substitution that the functions y=1+cos(x) and y=1+sin(x) are solutions of the non-homogeneous linear ODE.

$$y'' + y = 1$$

Initial Value Problem

- We are given the initial condition which is used to determine the arbitrary constant c in the general solution of the ODE. This result is a unique solution and known as a particular solution of the ODE.
- But in this case we will consider for second order homogeneous linear ODE an initial value problem consists of two initial conditions.
- That is $y_0(x_0) = k_0$, $y'(x_0) = k_1$ these conditions are used to determine arbitrary constant c_1 and c_2 in a general solution: $y=c_1y_1 + c_2y_2$.

Example #3

- Let's consider an example:

$$y'' + y = 0, \quad y(0) = 6.6, \quad y'(0) = -0.5$$

Definition of General Solution, Basis & Particular Solution

- A general solution of an ODE $y''+p(x)y'+q(x)y=0$ on an open interval I is a solution in which y_1 and y_2 are solutions of $y''+p(x)y'+q(x)y=0$ on I that are not proportional and c_1 and c_2 are arbitrary constants.
- These y_1 and y_2 are called basis of solution $y''+p(x)y'+q(x)y=0$ on I.
- A particular solution of $y''+p(x)y'+q(x)y=0$ on I is obtained if we assign specific values to c_1 and c_2.

Basis of a Solution

- If we say that y_1 and y_2 are proportional on I if for all x on I, i.e., $y_1 = ky_2 \rightarrow (a) \quad or \quad y_2 = ly_1 \rightarrow (b)$
- Where k and l are numbers, zero or not, and (a) implies (b) if and only if k≠0.
- We say that the functions y_1 and y_2 are linearly independent on an interval where they are defined if:

$$k_1 y_1(x) + k_2 y_2(x) = 0$$

- Everywhere on I implies $k_1=0$ and $k_2=0$.

Basis of a Solution

- And y_1 and y_2 are linearly dependent on I if above equation also holds for some constants k_1 , k_2 not both zero.
- Then if $k_1 \neq 0$ or $k_2 \neq 0$, we can divide and see that y_1 and y_2 are proportional,

$$y_1 = -\frac{k_2}{k_1} y_2 \quad or \quad y_2 = -\frac{k_2}{k_1} y_1$$

- Hence, a basis of solutions of $y'' + p(x)y' + q(x)y = 0$ on an open interval I is a pair of linearly independent solution on I.

Example #4

- Verify by substitution that $y_1=e^x$ and $y_2= e^{-x}$ are solutions of the ODE $y''- y=0$, solve for initial value problem $y(0)=6$ and $y'(0)=-2$.

The End