LECTURE # 6

: SOLVED EXAMPLES :-

EXAMPLE # 1 :-

$y_1 = \cos(x)$ and $y_2 = \sin x$ are the solutions of the homogeneous ODE $y'' + y = 0$ for all x. Verify by substitution.

SOL:-

Let's check $y_1 = \cos(x)$

$y_1' = -\sin x \quad , \quad y_1'' = -\cos x$

$y'' + y = 0$

$-\cos x + \cos x = 0$

$0 = 0$ hence, proved.

Now check $y_2 = \sin x$

$y_2' = \cos x \quad , \quad y_2'' = -\sin x$

$y'' + y = 0$

$-\sin x + \sin x = 0$

$0 = 0$ hence, proved

As y_1 and y_2 are the solutions of the given ODE the general solution is :-

$$y(x) = c_1 y_1 + c_2 y_2$$

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

EXAMPLE # 2:-

$$y_1 = 1+\cos x \quad \text{and} \quad y_2 = 1+\sin x$$

are solutions of non-homogeneous ODE

$$y''+y=1$$

Sol:-

let's check both the solutions one by one.

$$y_1' = -\sin x \;,\; y_1'' = -\cos x$$

$$y''+y=1$$
$$-\cancel{\cos x}+1+\cancel{\cos x}=1$$
$$1=1$$ hence proved.

Now, $y_2' = \cos x \;,\; y_2'' = -\sin x$

$$y''+y=1$$
$$-\cancel{\sin x}+1+\cancel{\sin x}=1$$
$$1=1$$ hence proved.

EXAMPLE # 3:-

$$y''+y=0 \quad , y(0)=3.0 \; , y'(0)=-0.5.$$

$y(x) = c_1\cos(x) + c_2\sin(x)$ general solution.

particular solution = ?

Sol:-

$$y'(x) = -c_1\sin(x) + c_2\cos(x)$$

$$y(0) = c_1\cos(0) + c_2\sin(0) = 3.0$$
$$c_1(1) + c_2(0) = 3.0$$
$$c_1 = 3.0$$

$$y(0) = -c_1\sin(0) + c_2\cos(0) = -0.5$$
$$-c_1(0) + c_2(1) = -0.5$$
$$c_2 = -0.5$$

hence the particular solution is,

$$y(x) = 3.0\cos(x) - 0.5\sin(x).$$

Example # 4:

Proof $y = e^{rx}$ is a solution of $y'' + 5y' + 6y = 0$. Also find its particular solution using $y(0) = 2$, $y'(0) = 3$

Sol:

$$y = e^{rx}$$
$$y' = re^{rx}, \quad y'' = r^2 e^{rx}$$
$$y'' + 5y' + 6y = 0$$

$$r^2 e^{rx} + 5re^{rx} + 6e^{rx} = 0 \rightarrow (1)$$
$$e^{rx}(r^2 + 5r + 6) = 0$$
$$r^2 + 5r + 6 = 0 \Rightarrow \text{characteristic equ.}$$
$$(r+2)(r+3) = 0$$
$$r+2 = 0, \quad r+3 = 0$$
$$r = -2, \quad r = -3$$

$$y_1 = e^{-2x}, \quad y_2 = e^{-3x}$$

$$y(x) = c_1 e^{-2x} + c_2 e^{-3x} \Rightarrow \text{general solution.}$$

Let's proof.

$$r = -2$$
$$y_1 = e^{-2x} \text{ put in equ (1)}$$
$$r^2 e^{rx} + 5re^{rx} + 6e^{rx} = 0$$
$$(-2)^2 e^{-2x} + 5(-2)e^{-2x} + 6e^{-2x} = 0$$
$$4e^{-2x} - 10e^{-2x} + 6e^{-2x} = 0$$
$$-6e^{-2x} + 6e^{-2x} = 0$$

$0 = 0$ hence proved.

Now put $r = -3$ in equ ①

$$r^2 e^{rx} + 5 r e^{rx} + 6 e^{rx} = 0$$
$$(-3)^2 e^{-3x} + 5(-3) e^{-3x} + 6e^{-3x} = 0$$
$$9e^{-3x} - 15e^{-3x} + 6e^{-3x} = 0$$
$$\cancel{-6e^{-3x}} + \cancel{6e^{-3x}} = 0$$
$0 = 0$ hence proved.

Now for particular solution:

$$y(x) = c_1 e^{-2x} + c_2 e^{-3x}$$

$$y'(x) = -2c_1 e^{-2x} - 3c_2 e^{-3x}$$

$$y(0) = c_1 e^{-2(0)} + c_2 e^{-3(0)} = 2 \rightarrow ②$$
$$y'(0) = -2c_1 e^{-2(0)} - 3c_2 e^{-3(0)} = 3 \rightarrow ③$$

$$c_1 + c_2 = 2 \rightarrow ②$$
$$-2c_1 - 3c_2 = 3 \rightarrow ③$$

Multiply 2 by equ ② and add with equ ③

$$2(c_1 + c_2 = 2)$$

$$\cancel{2c_1} + 2c_2 = 4$$
$$\cancel{-2c_1} - 3c_2 = 3$$
$$-c_2 = 7$$
$$\boxed{c_2 = -7}$$ put in equ ②

$$c_1 + c_2 = 2$$
$$c_1 - 7 = 2$$
$$\boxed{c_1 = 9}$$

hence,

$$y(x) = 9e^{-2x} - 7e^{3x}$$ particular solution.