Calculus-II

Lecture #7

Homogeneous Linear ODEs

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Homogeneous Linear ODEs with Constant Cofficients

ODEs with Constant Coefficients

- Let's consider second order homogeneous linear ODEs whose coefficients a and b are constant:

$$y'' + ay' + by = 0 \rightarrow (1)$$

- The solution of the first order linear ODE with constant coefficient k is: y'+ky=0, is an exponential function:

$$y = ce^{-kx}$$

- Now let's try it as a solution of the function for equ: (1)

$$y = e^{\lambda x} \rightarrow (2)$$

ODEs with Constant Coefficients (cont.)

$$y' = \lambda e^{\lambda x} \quad and \quad y'' = \lambda^2 e^{\lambda x}$$

- Substitute in equ (1):

$$y'' + ay' + by = 0$$
$$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0$$
$$e^{\lambda x}\left(\lambda^2 + a\lambda + b\right) = 0$$

- Hence if λ is a solution of the characteristic equation or auxiliary equation i.e., $\lambda^2 + a\lambda + b$
- Then the exponential function in equ(2) is a solution of the ODE in equ (1).

ODEs with Constant Coefficients (cont.)

- From algebra the roots of the quadratic equation are:

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right), \quad \lambda_1 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right)$$

- The derivation shows that $y_1 = e^{\lambda_1 x} \quad and \quad y_2 = e^{\lambda_2 x}$ are solutions of equ (1).

Example #1

- Solve:

$$y'' - 5y' + 6y = 0$$

Cases

- The quadratic equation may have three kinds of roots, depending on the sign of the discriminant $a^2 - 4b$, namely:
- **Case I:** Two real roots if $a^2 - 4b > 0$.
- **Case II:** A real double root if $a^2 - 4b = 0$.
- **Case III:** Complex Conjugate roots if $a^2 - 4b < 0$.

Case I

- A basis of solution of equ (1) on any interval is:

$$y_1 = e^{\lambda_1 x} \quad and \quad y_2 = e^{\lambda_2 x}$$

- Because y_1 and y_2 are defined for all x and their quotient is not constant. The corresponding general solution is:

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

Example #2

- Solve the initial value problem:

$$y'' + y' - 2y = 0; \quad y(0) = 4, \quad y'(0) = -5$$

Case II

- If the discriminant $a^2 - 4b$ is zero, then we get only one root, $\lambda = \lambda_1 = \lambda_2 = -a/2$, hence only one solution: $y_1 = e^{-(a/2)x}$
- To obtain a second independent solution y_2, we use the method of reduction of order.
- Setting $y_2 = uy_1$, substituting this and its derivatives in equ (1) we have:

$$\left(u''y_1 + 2u'y_1' + uy_1''\right) + a\left(u'y_1 + uy_1'\right) + buy_1 = 0$$

- Collecting terms in u'', u', and u we obtain:

$$u''y_1 + u'\left(2y_1' + ay_1\right) + u\left(y_1'' + ay_1' + by_1\right) = 0$$

Case II (cont.)

- The expression in the last parenthesis is zero, since y_1 is a solution of equ (1). The expression in the first parenthesis is zero too, since: $2y_1' = -ae^{-ax/2} = -ay_1$
- We are thus left with $u''y_1=0$. Hence $u''=0$.
- By two integrations, $u=c_1x+c_2$. To get a second independent solution $y_2=uy_1$, we can simply choose $c_1=1$ and $c_2=0$ and take $u=x$.
- Then $y_2=xy_1$. Since these solutions are proportional, they form a basis.

Case II (cont.)

- Hence in the case of double root a basis of solution of equ (1) on any interval is:

$$e^{-ax/2}, xe^{-ax/2}$$

- The corresponding general solution is:

$$y = \left(c_1 + c_2 x\right) e^{-ax/2}$$

Example #3

- Solve the initial value problem:

$$y'' + y' + 0.25y = 0; \quad y(0) = 3, \quad y'(0) = -3.5$$

Case III

- This case occurs if the discriminant $a^2 - 4b$ of the characteristic equation is negative.
- The roots are the complex $\lambda = -\frac{1}{2}a \pm i\omega$ that gives the complex solutions of the ODE.
- However, we will show that we can obtain a basis of real solutions: $y_1 = e^{-ax/2}\cos\omega x, \quad y_2 = e^{-ax/2}\sin\omega x$
- Where $\omega^2 = b - a^2 1/4$.
- Hence a real general solution in Case III is:

$$y = e^{-ax/2}\left(A\cos wx + B\sin wx\right)$$

Example #4

- Solve the initial value problem:

$$y'' + 0.4y' + 9.04y = 0$$

Summary of Cases I-III

Case	Roots of (2)	Basis of (1)	General Solution of (1)
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Real double root $\lambda = -\frac{1}{2}a$	$e^{-ax/2}, xe^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$
III	Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega$, $\lambda_2 = -\frac{1}{2}a - i\omega$	$e^{-ax/2} \cos \omega x$ $e^{-ax/2} \sin \omega x$	$y = e^{-ax/2}(A \cos \omega x + B \sin \omega x)$

Derivation in Case III

- If we skip the systematic derivation of these real solutions i.e., $y_1 = e^{-ax/2}\cos\omega x, \quad y_2 = e^{-ax/2}\sin\omega x$ by means of the complex exponential function e^z of a complex variable z=r+it.
- We write r+it and not x+iy because x and y occur in the ODE.
- The definition of e^z in terms of the real functions e^r, cos t and sin t is:

$$e^z = e^{r+it} = e^r e^{it} \Rightarrow e^r\left(\cos t + i\sin t\right)$$

Derivation in Case III (cont.)

- For real z=r, hence t=0, cos 0 =1, sin 0 =0, we get the real exponential function e^r.
- $$e^{z_1+z_2} = e^{z_1}e^{z_2}$$ also real and correct.
- If we use Maclaurin series of e^z with z=it as well as i^2=-1, e^3=-I, i^4=1 ….**etc and reorder the terms as shown**:

$$e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \ldots\ldots$$

$$= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \ldots. + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots\ldots.\right)$$

Derivation in Case III (cont.)

- e^{it}=cos t + I sin t also called Euler Formula.
- Multiplying it with e^r gives:

$$e^z = e^{r+it} = e^r e^{it} \Rightarrow e^r\left(\cos t + i\sin t\right)$$

- We note that e^{-it} = cos(-t) + isin (-t)=cos t – i sin t, so that by addition and subtraction we get:

$$\cos t = \frac{1}{2}\left(e^{it} + e^{-it}\right), \quad \sin t = \frac{1}{2i}\left(e^{it} - e^{-it}\right)$$

- In case III the radicand $a^2 - 4b$ is negative. Hence $4b - a^2$ is positive and using √-1=I, we obtain:

Derivation in Case III (cont.)

$$\frac{1}{2}\sqrt{a^2-4b}=\frac{1}{2}\sqrt{-\left(4b-a^2\right)}=\sqrt{-\left(b-\frac{1}{4}a^2\right)}=i\sqrt{b-\frac{1}{4}a^2}=i\omega$$

- And: $$\lambda_1=\frac{1}{2}a+i\omega \quad and \quad \lambda_2=\frac{1}{2}a-i\omega$$

- Using r = -(1/2)ax and t=ωx, we thus obtain:

$$e^{\lambda_1 x}=e^{-(a/2)x+i\omega x}=e^{-(a/2)x}\left(\cos\omega x+i\sin\omega x\right)$$

$$e^{\lambda 2x}=e^{-(a/2)x-i\omega x}=e^{-(a/2)x}\left(\cos\omega x-i\sin\omega x\right)$$

Example #5

- Solve the initial value problem:

$$20y'' + 4y' + y = 0, \quad y(0) = 3.2, \quad y'(0) = 0$$

Example #6

- Solve the following ODE:

$$y'' - 6y' - 7y = 0$$

The End