Calculus-II

Lecture #8

Differential Operators

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Differential Operators

Differential Operators

- Operator is a transformation that transforms a function into another function.
- The Differential operator D is an operator which transforms a function into its derivative.
- In operator notation we write $D=d/dx$ and $Dy = y' = dy/dx$.
- Similarly, for the higher derivatives we write $D^2y = D(Dy) = y''$ and so on.
- For a homogeneous linear ODE $y''+ay'+by=0$ with constant coefficients we can now introduce the second-order differential operator: $L = P(D) = D^2+aD+bI$
- Where I is the identity operator defined by $Iy=y$.

Differential Operators (cont.)

- Then we can write that ODE as:

$$Ly = P(D)y = (D^2 + aD + bI)y = 0$$

- P suggests "polynomial" and L is a linear operator.
- Since:

$$(De^{\lambda})(x) = \lambda e^{\lambda x} \quad and \quad (D^2 e^{\lambda})(x) = \lambda^2 e^{\lambda x} \quad we \quad obtain$$

$$Le^{\lambda}(x) = P(D)e^{\lambda}(x) = (D^2 + aD + bI)e^{\lambda}(x)$$

$$= (\lambda^2 + a\lambda + b)e^{\lambda x} = P(\lambda)e^{\lambda x} = 0$$

Differential Operators (cont.)

- $e^{\lambda x}$ is a solution of the ODE if and only if λ is a solution of the characteristic equation $P(\lambda)=0$.

Example #1

- Factor $P(D) = D^2 - 3D - 40I$ and solve $P(D)y=0$.

Example #2

- Factor and solve the following:

$$\left(D^2 + 4.00D + 3.36I\right)y = 0$$

Euler-Cauchy Equations

Euler-Cauchy Equations

- Euler-Cauchy equations are ODEs of the form:

$$x^2 y'' + axy' + by = 0$$

- With given constants a and b and unknown function y(x).
- We substitute: $y=x^m$, $y'=mx^{m-1}$, $y''=m(m-1)x^{m-2}$ into the above equation. This gives:

$$x^2 m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0$$

- We now see that $y=x^m$ was a rather natural choice because we have obtained a common factor x^m.

Euler-Cauchy Equations (cont.)

- Dropping it, we have the auxiliary equation: m(m-1)+am+b=0 or

$$m^2 + (a-1)m + b = 0$$

- Hence $y = x^m$ is a solution of Euler-Cauchy equation if and only if m is a root of above equation.

Case I

- Real different roots m_1 and m_2 give two real solutions:

$$y_1(x) = x^{m_1} \quad and \quad y_2(x) = x^{m_2}$$

- These are linearly independent since their quotient is not constant.
- Hence they constitute a basis of solution of Euler-Cauchy equation for all x which they are real.
- The corresponding general solution is:

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

Case II

- A real double root $m_1=1/2(1-a)$ occurs if and only if $b=1/4(a-1)^2$.
- The Euler-Cauchy equation then has the form:

$$x^2 y'' + axy' + \frac{1}{4}(1-a)^2 y = 0$$

- $y_1=x^{(1-a)/2}$
- To obtain a second linearly independent solution we apply the method of reduction of order.

Case II (cont.)

- Hence for all x for which y_1 and y_2 are defined are real, a general solution is: $y = \left(c_1 + c_2 \ln x\right) x^m$

Example #3

- Solve:

$$x^2 y'' - 5xy' + 9y = 0$$

Case III

- Complex conjugate roots are of minor practical importance.

Example #4

- Solve:

$$x^2 y'' + 0.6xy' + 16.04y = 0$$

Example #5

- Find a general solution. Show the details of your work.

$$x^2 y'' - 6y = 0$$

Example #6

- Find the IVP. Show the details of your work.

$$x^2 y'' - 4xy' + 6y = 0, \quad y(1) = 1, \quad y'(1) = 0$$

The End