Example #1

$$y'' + y = 0.001x^2, \quad y(0) = 0, \quad y'(0) = 1.5$$

Solution: Step 1:- General solution of the homogeneous ODE.

$$y'' + y = 0$$
$$\lambda^2 + 1 = 0$$
$$\lambda = \pm i\sqrt{1}$$
$$y_h = A\cos x + B\sin x.$$

Step 2:- Solution y_p of the non homogeneous ODE.

We first try $y_p = Kx^2$.

$y_p'' = 2K$ Now substitute in ODE

$$2K + Kx^2 = 0.001x^2$$

For this to hold for all x, the coefficients of each power of x (x^2 and x^0) must be the same on both sides. Thus.

$$K = 0.001$$
$$2K = 0, \text{ a contradiction}$$

The second line in Table 2.1 suggest the choice.

$y_p = K_2x^2 + K_1x + K_0$ Then

$\cancel{y_p'' = 2K_2 +}$

Then $y_p'' + y_p = 2K_2 + K_2x^2 + K_1x + K_0 = 0.001x^2$

Equating the coefficients of x^2, x and x^0 on both sides.

$$K_2 = 0.001, \quad K_1 = 0, \quad 2K_2 + K_0 = 0$$
$$K_0 = -2K_2 \Rightarrow -2(0.001) \Rightarrow -0.002.$$

This gives $y_p = 0.001x^2 - 0.002$

$$y = y_h + y_p = A\cos x + B\sin x + 0.001x^2 - 0.002.$$

Step 3: Initial value problem.

$\therefore y(0) = 0$

$y(0) = A\cos(0) + B\cancel{\sin(0)} + 0.001\cancel{(0)} - 0.002$

$y(0) = A - 0.002 \Rightarrow 0.$

$A = 0.002.$

$\therefore y'(0) = 1.5.$

$y'(0) = -A\cancel{\sin x} + B\cos x + 0.\cancel{002}x$

$y'(0) = B = 1.5.$

$$y = 0.002\cos x + 1.5\sin x + 0.001x^2 - 0.002.$$

Example #2

Example 2: Application ...

$y'' + 3y' + 2.25y = -10e^{-1.5x} \qquad y(0) = 1,\ y'(0) = 0$

Step 1: General solution of the homogeneous ODE.

$\lambda^2 + 3\lambda + 2.25 = 0$

$(\lambda + 1.5)^2 = 0$

$\lambda = -1.5$ Double root.

$y_h = (c_1 + c_2x)e^{-1.5x}$

Step 2: Solution y_p of the non-homogeneous ODE.

We will multiply our choice by x^2 because we have double root solution of the homogeneous ODE.

$y_p = Ce^{-1.5x}x^2$

$y'_p = C[2x - 1.5x^2]e^{-1.5x}$

$y''_p = C[2 - 3x - 3x + 2.25x^2]e^{-1.5x}$

Omit $e^{-1.5x}$ and substitute.

$C[2-6x+2.25x^2] + 3C[2x-1.5x^2] + 2.25Cx^2 = -10$

Comparing the coefficients of x^2, x, x^0 gives.

$0=0, \quad 0=0, \quad 2C=-10$

$C=-5$

This gives the solution of $y_p = -5x^2e^{-1.5x}$.

general solution of ODE will be.

$y = y_h + y_p$

$y = (c_1+c_2x)e^{-1.5x} - 5x^2e^{-1.5x}$.

Step 3:- I.V.P.

$y(0)=0$.

$y = c_1e^0 + c_2(0)e^0 - 5(0)e^0$

$y = c_1 = 1$

$y'(0) = 1.5$

$y' = (-1.5c_1 + c_2 - 1.5c_2x)e^{-1.5x} - 10xe^{-1.5x} + 7.5x^2e^{-1.5x}$

$y'(0) = -1.5c_1e^0 + c_2e^0 - 1.5c_2(0)e^0 - 10(0)e^0 + 7.5(0)e^0$

$y' = c_2 - 1.5c_1 = 0.$

$c_2 - 1.5c_1 = 0$

$c_2 = 1.5$.

$y = (1+1.5x)e^{-1.5x} - 5x^2e^{-1.5x}$. particular sol

Example #3

$$y'' + 2y' + 5y = e^{0.5x} + 40\cos 10x - 190\sin 10x$$

$y(0) = 0.16$, $y'(0) = 40.0$

Solution :- Step 1 General Solution of Homogeneous ODE.

$$\lambda^2 + 2\lambda + 5 = 0.$$

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} \Rightarrow \frac{-2 \pm \sqrt{-16}}{2}$$

$$\lambda = -\frac{2}{2} \pm i\frac{4}{2} \Rightarrow -1 \pm i2.$$

$$y_h = e^{-x}[A\cos 2x + B\sin 2x]$$

Step 2:- Solution of the non-homogeneous ODE.

$y_p = y_{p_1} + y_{p_2}$

$y_{p_1} \Rightarrow$ corresponds to the exponential term.

$y_{p_2} \Rightarrow$ sum of other two terms.

$y_{p_1} = Ce^{0.5x}$ $\quad y_{p_1}' = 0.5\,Ce^{0.5x}$ $\quad y_{p_1}'' = 0.25\,Ce^{0.5x}$

Substitute in given ODE and omit exponential factor gives:-

$$0.25C + 2(0.5)C + 5C = 1$$

$$C = 0.16$$

$$y_{p_1} = 0.16e^{0.5x}.$$

$y_{p_2} = k\cos 10x + M\sin 10x$

$y_{p_2}' = -10k\sin 10x + 10M\cos 10x$

$y_{p_2}'' = -100k\cos 10x - 100M\sin 10x$

substitute in ODE and

$-100k + 2(10)M + 5k = 40 \Rightarrow 20M - 95k = 40$

$-100M - 10k(2) + 5M = -190 \Rightarrow -20k - 95M = -190$

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$\cancel{95}K = \frac{-40+20M}{95}$ put in (b)

$-20\left[\frac{-40+20M}{95}\right] - 95M = -190$

Multiply 95 on both sides

$-20[-40+20M] - 9025M = \cancel{-190}\ -18050$

$+800 - 400M - 9025M = \cancel{-190}\ -18050$

$\cancel{-9428M \Rightarrow 880 - 17250}$

$\cancel{M \Rightarrow}$

$\cancel{-}9425M \Rightarrow \cancel{-}18850$

$M \Rightarrow 2$ put in (a)

$-95K + 20(2) = 40$

$K=0 \Rightarrow y_{p_2} = 2\sin 10x.$

$y = y_h + y_{p_1} + y_{p_2}$

$= e^{-x}(A\cos 2x + B\sin 2x) + 0.16e^{0.5x} + 2\sin 10x.$

Step 3 I.V.P :-

$y(0) = 0.16$

$y = e^{0}[A\cos(0) + B\cancel{\sin(0)}] + 0.16e^{0} + 2\cancel{\sin(0)}$

$y = A + 0.16 = 0.16$

$A = 0.$

$y' = e^{-x}[-A\cos 2x - B\sin 2x - 2A\sin 2x + 2B\cos 2x] + 0.08e^{0.5x} + 20\cos 10x.$

$y'(0) = -A + 2B + 0.08 + 20 = 40.08$

$B = 10.$

$y = 10e^{-x}\sin 2x + 0.16e^{0.5x} + 2\sin 10x.$

Example #4

7). $y'' + 6y' + 73y = 80e^{x} \cos 4x.$

Step 1:- General solution for homogeneous ODE.

$$y'' + 6y' + 73y = 0$$

$$r^2 + 6r + 73 = 0$$

$$r = \frac{-6 \pm \sqrt{36 - 292}}{2} \Rightarrow \frac{-6 \pm i\sqrt{256}}{2}$$

$$r = \frac{-6}{2} \pm \frac{i16}{2} \Rightarrow -3 \pm i8.$$

$$y_h = e^{-3x}(A\cos 8x + B\sin 8x).$$

Step 2: Find y_p. as its a product. we will first deal with $\cos 4x$ and then with e^x.

$$y_{p_1} = A\cos 4x + B\sin 4x.$$

Now simply multiply e^x gives.

$$y_p = e^x A\cos 4x + e^x B\sin 4x.$$

$$y_p' = Ae^x\cos 4x - 4Ae^x \sin 4x + Be^x\sin 4x + 4e^x B\cos 4x.$$

$$y_p'' = Ae^x\cos 4x - 4e^x A\sin 4x - 4Ae^x\sin 4x - 16Ae^x\cos 4x$$
$$+ Be^x\sin 4x + 4Be^x\cos 4x + 4Be^x\cos 4x - 16Be^x\sin 4x.$$

$$y_p'' = (-15Ae^x\cos 4x - 8e^x A\sin 4x) + (-15Be^x\sin 4x + 8Be^x\cos 4x)$$

substitute it in given ODE.

$$(-15Ae^x\cos 4x - 8e^x A\sin 4x) + (-15Be^x\sin 4x + 8Be^x\cos 4x) + 6(Ae^x\cos 4x$$
$$- 4Ae^x\sin 4x + Be^x\sin 4x + 4e^x B\cos 4x) + 73(e^x A\cos 4x + e^x B\sin 4x)$$
$$= 80e^x\cos 4x.$$

$$-15Ae^x\cos 4x - 8e^x A\sin 4x - 15Be^x\sin 4x + 8Be^x\cos 4x + 6Ae^x\cos 4x$$
$$- 24Ae^x\sin 4x + 6Be^x\sin 4x + 24e^x B\cos 4x + 73e^x A\cos 4x + 73e^x B\sin 4x$$
$$= 80e^x\cos 4x.$$

$$64Ae^x\cos 4x - 32e^x A\sin 4x + 64Be^x\sin 4x + 32e^x B\cos 4x$$
$$= 80e^x\cos 4x.$$

Now comparing coefficients of sin and cos

$\cos 4x,\quad 64A + 32B = 80 \rightarrow (a)$, $32B = 80 - 64A$

$B = \frac{80}{32} - \frac{64A}{32} \Rightarrow 2.5 - 2A$. put in (b)

$\sin 4x,\quad -32A + 64B = 0 \rightarrow (b)$

$$-32A + 64(2.5 - 2A) = 0$$

$$-32A + 160 - 128A = 0$$

$$-160A = -160$$

$A = +1$. put in (a)

$$64A + 32B = 80$$

$$+64 + 32B = \cancel{80}\ 80$$

$$B \Rightarrow \cancel{\;}\ 0.5$$

$$y_p = e^{x}\left[-\cos 4x + 0.5\sin 4x\right)$$

$$y = y_h + y_p \Rightarrow e^{-3x}(A\cos 8x + B\sin 8x) + e^{x}(+\cos 4x + 0.5\sin 4x)$$

Example #5

$$y'' + y = \sec x = \frac{1}{\cos x}.$$

Solution:

A basis of solution of the homogeneous ODE is $y_1 = \cos x$, $y_2 = \sin x$. This gives the Wronskian

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$
$$= \cos x \cos x - \sin x(-\sin x)$$
$$= \cos^2 x + \sin^2 x \Rightarrow 1 \quad \text{Identity}$$

From (2) choosing zero constants of integration we get particular solution

$$y_p = -y_1 \int \frac{y_2 r}{W}\, dx + y_2 \int \frac{y_1 r}{W}\, dx \rightarrow (2)$$

$$= -\cos x \int \sin x \sec x \, dx + \sin x \int \cos x \sec x \, dx$$

$$= -\cos x \int \sin x \cdot \frac{1}{\cos x}\, dx + \sin x \int \cancel{\cos x}\, \frac{1}{\cancel{\cos x}}\, dx$$

$$= -\cos x \left[-\ln|\cos x|\right] + x \sin x.$$

$$y_p = \cos x \ln|\cos x| + x \sin x.$$

$$y = y_h + y_p$$

$$y = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$$

$$y = \left[c_1 + \ln|\cos x|\right] \cos x + \left[c_2 + x\right] \sin x.$$

Example #6

5) $y'' + y = \tan x. \Rightarrow y'' + y = \frac{\sin x}{\cos x}$

Step 1: general solution of h

$y'' + y = 0$

$\lambda^2 + 1 = 0$

$\lambda = \frac{0 \pm \sqrt{0-4}}{2}$

$\lambda = \frac{0 \pm \sqrt{-4}}{2}$

$\lambda_1 = \frac{+i\cancel{2}}{\cancel{2}} = i$

$\lambda_2 = -i$

$\cos x, \sin x$

$y_1 = \cos x, \quad y_2 = \sin x$

$y_h = A\cos x + B\sin x.$

$W = y_1 y_2' - y_2 y_1'$

$\Rightarrow \cos x(\cos x) - \sin x(-\sin x) \Rightarrow 1.$

Now for y_p:-

$$y_p = -y_1 \int \frac{y_2 b}{W} dx + y_2 \int \frac{y_1 b}{W} dx$$

$$= -\cos x \int \sin x \left(\frac{\sin x}{\cos x}\right) dx + \sin x \int \cos x \left(\frac{\sin x}{\cos x}\right) dx$$

$$= -\cos x \int \sin x \left(\frac{\sin x}{\cos x}\right) dx + \sin x \int \sin x \, dx.$$

let's consider $\int \sin x \tan(x) \, dx.$

apply integration by parts.

$\int uv' = uv - \int u'v$

$u = \tan(x), \quad u' = \sec^2 x, \quad v' = \sin x, \quad v = -\cos x.$

$$\int \sin x \tan x \, dx = (\tan x)(-\cos x) - \int \sec^2 x(-\cos x) \, dx.$$

$$= \left(\frac{\sin x}{\cos x}\right)(-\cos x) - \int \frac{1}{\cos^2 x}(-\cos x) \, dx$$

$$= -\sin x + \int \frac{1}{\cos x} dx.$$

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Now. $\frac{1}{\cos x} = \sec x.$

$$= -\sin(x) + \int \sec x \, dx.$$

$$= -\sin(x) + \ln|\sec x + \tan x|.$$

$$y_p = -\cos x \left[-\sin(x) + \ln|\sec x + \tan x|\right] + \sin x \, (-\cos x)$$

$$y_p = +\cos(x)\sin(x) - \cos(x)\ln|\sec x + \tan x| - \sin(x)\cos(x)$$

$$y = y_h + y_p$$

$$y = A\cos x + B\sin x + \cos(x)\sin(x) - \cos(x)\ln|\sec x + \tan x|$$
$$- \sin(x)\cos(x).$$ ✓