Calculus-II

Lecture #9

Non-homogeneous ODEs

21st May, 2019

Nonhomogeneous ODEs

Nonhomogeneous

- Consider the second-order nonhomogeneous linear ODE:

$$y'' + p(x)y' + q(x)y = r(x) \rightarrow (1)$$

- Where $r(x) \neq 0$.
- The general solution for the above ODE is the sum of a general solution of the corresponding homogeneous ODE and a particular solution of above equation.

General & Particular Solution

- A general solution of the nonhomogeneous ODE on an open interval is a solution of the form:

$$y(x) = y_h(x) + y_p(x)$$

- Here $y_h = c_1y_1 + c_2y_2$ is a general solution of the homogenous ODE on I and y_p is any solution of (1) on I containing no arbitrary constants.
- A particular solution of (1) on I is a solution obtained from above equation by assigning specific values to the arbitrary constants c_1 and c_2.

Method of Undetermined Coefficients

- To find the solution y_p of (1) we use the method of undetermined coefficients and other method is variation of parameters.
- Lets discuss the method of Undetermined coefficients.
- This method is suitable for linear ODE with constant coefficients a and b.

$$y'' + ay' + by = r(x) \rightarrow (2)$$

- Where r(x) is an exponential function, a power of x, a cosine or sine, or a sum or product of such functions.

Method of Undetermined Coefficients (cont.)

- We choose a form for y_p similar to $r(x)$ but with unknown coefficients to be determined by substituting that y_p and its derivatives into the ODE.

Choice Rules

- **Basic Rule**: If r(x) in equ (2) is one of the fucntions in the first column in the following table, choose y_p in the same line and determine its undetermined coefficients by substituting y_p and its derivatives in to (2). This rule applies when r(x) is a single term.

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

Example #1

- **Application of the Basic rule.**
- Solve the initial value problem:

$$y'' + y = 0.001x^2, \quad y(0) = 0, \quad y'(0) = 1.5$$

Choice Rules (cont.)

<u>Modification Rule</u>: If a term in your choice of y_p happens to be a solution of the homogeneous ODE corresponding to (2), multiply this term by x or by x^2 if this solution corresponds to the double root of the characteristic equatiion of the homogeneous ODE.

Example #2

- **Application of the Modification rule.**
- Solve the initial value problem:

$$y'' + 3y' + 2.25y = -10e^{-1.5x}, \quad y(0) = 1, \quad y'(0) = 0$$

Choice Rules (cont.)

- **Sum Rule**: If r(x) is a sum of functions in the first column of following table, choose for y_p the sum of the function in the corresponding lines of the second column.

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

Example #3

- **Application of the Sum rule.**
- Solve the initial value problem:

$$y'' + 2y' + 5y = e^{0.5x} + 40\cos 10x - 190\sin 10x, \quad y(0) = 0.16, \quad y'(0) = 40.08$$

Example #4

- Find a general solution of the following ODE:

$$y'' + 6y' + 73y = 80e^{x}\cos 4x$$

Variation of Parameters

Solution by Variation of Parameters

- As we have nonhomogeneous linear ODE:

$$y'' + p(x)y' + q(x)y = r(x) \rightarrow (1)$$

- To obtain y_p when r(x) of not too complicated, we can use the method of undetermined coefficients.
- This method is restricted to functions r(x) whose derivatives are of a form similar to r(x) itself, it is desirable to have a method valid for more general ODE. Its called the method of variation of parameters.
- Hence p, q and r in (1) may be variable but we assume that they are continuous on some open interval I.

Solution by Variation of Parameters (cont.)

- Lagrange's method gives a particular solution y_p on I in the form:

$$y_p(x) = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx \rightarrow (2)$$

- Where y_1 and y_2 form a basis of solutions of the corresponding homogeneous ODE:

$$y'' + p(x) y' + q(x) y = 0 \rightarrow (3)$$

- And W is the Wronskian of y_1 and y_2.

$$W = y_1 y_2' + y_2 y_1' \rightarrow (4)$$

Example #5

- Solve the nonhomogeneous ODE:

$$y'' + y = \sec x = \frac{1}{\cos x}$$

Example #6

- Solve the nonhomogeneous ODE:

$$y'' + y = \tan x = \frac{\sin x}{\cos x}$$

The End